Gradient Estimation on Lagrangian Data Using Equations of Motion as Physical Constraints

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ABSTRACT

The advent of high-density particle tracking velocimetry has dramatically improved both spatial and temporal resolution available to researchers. However, making full use of the temporal data along a pathline does not yet exist. Here, a novel method for estimating gradients is proposed by applying equations of motion as constraints on their values, while leveraging the substantial derivative along a pathline. Specifically, here we use the vorticity transport equation to limit the possible values of velocity-gradient components. It is shown that the method, although preliminary, can reduce the noise in velocity gradients along a pathline and throughout a flow field.

1. Introduction

Due to the ubiquity of particle image velocimetry (PIV), optical measurements have traditionally been used to acquire Eulerian flow-field data on a spatially uniform grid. In contrast, particle tracking velocimetry (PTV) produces Lagrangian flow-field data on a spatially non-uniform grid. The advent of novel tracking algorithms such as “Shake-The-Box” has now made it possible to achieve spatial resolutions with 4D-PTV that are greater than Tomographic PIV [13]. With the spatial resolution of 4D-PTV now matching that of Tomo-PIV, it is reasonable to expect the wider adoption of PTV throughout the fluid-mechanics community.

Collecting data in a Lagrangian frame presents with two significant advantages: it provides direct access to the material derivative and to the flow map. Directly measuring the material derivative has far-reaching applications. One such application is pressure extraction via the Poisson equation. In a Lagrangian frame, this operation can be performed accurately, since the material derivative of velocity is directly measured. In contrast, if pressure is extracted from the material derivative in an Eulerian frame, one must be conscious of the possibility of compounding errors from the multiplication and summation of the many necessary measured terms [9]. Another application is in the study of entrainment across rotational non-rotational interfaces, since it is common practice
to calculate the velocity of entrained fluid across the interface by normalizing the material derivative of enstrophy by the magnitude of the enstrophy gradient [15]. Finally, having direct access to the flow map is of particular use when calculating Lagrangian coherent structures [12], which are frame-invariant, user threshold-independent manifolds that separate a flow field into regions of similar flow-map strain [9]. With Eulerian data, one is limited to Eulerian coherent structures, which are both frame-variant and threshold-dependent [5]. One can attempt to acquire the flow map from PIV data by synthetically seeding measured velocity fields with tracers and integrating to determine their paths. However, this method is limited due to the truncation error and is computationally expensive [10].

However, PIV still offers one notable advantage over PTV, which is the ease and robustness of spatial differentiation due to its spatially uniform grid. The uniform spacing of data nodes allows one to quickly derive numerical representations of the spatial derivative, while the accuracy of the spatial derivative can be improved by using more data nodes surrounding a point of interest, which in turn reduces the residual error [2]. The uniform grid also simplifies spatial and temporal filtering, which can in turn reduce the effect of data outliers on spatial differentiation. In contrast, the spatial non-uniformity of the data produced by PTV, which changes from time-step to time-step, makes spatial differentiation computationally slow and difficult to implement. Currently, typical methods to calculate the spatial derivative at a point of interest include: residual-minimization techniques that use an overdetermined system of equations based on neighboring nodes from which the spatial gradient can be calculated [6, 7]; or weighted-average techniques that calculate the gradient as a sum of weighted predictions based on neighboring nodes [3]. Moreover, more sophisticated interpolation techniques make use of physical information about the flow, such as a divergence-free velocity field, in order to improve resolution [4, 14].

The ability to mitigate the effect of data outliers remains an issue when calculating the spatial gradient on non-uniform data. Enforcing Lipschitz continuity on the gradient is likely the simplest method, while restraining the gradient to exist only within certain bounds is another [1]. However, applying these constraints aggressively can adversely affect the gradient, and thus information of the gradient should be known a priori, which may be unavailable. Furthermore, even if these constraints are placed conservatively, their lack of physical basis calls the accuracy of the resulting gradient into question. Thus, constraints that have some physical basis should be placed on the flow instead. For instance, it has been shown that tomographic PIV can be super-sampled below the Nyquist frequency by using the high spatial resolution to improve temporal resolution through
the Navier-Stokes equations [14]. Here, we propose to use the high temporal resolution of 4D-PTV to improve the quality of spatial gradients by constraining measurements through the vorticity-transport equation.

This method attempts to mitigate the effect of data outliers using physical constraints of incompressibility and high Reynolds numbers. To test the method, 4D-PTV “Shake-the-Box” data was collected from the wake of a NACA-0012 profile. The apparatus used to collect the data is presented herein. Finally, preliminary results of the above methodology are presented for the NACA-0012 wake.

2. Gradient-Evaluation Techniques

The current section begins by formulating the problem at hand, which is the evaluation of the spatial derivative from unstructured, Lagrangian data. Then, this section presents a gradient-estimation scheme that minimizes an objective function derived from the vorticity-transport equation.

Consider a set of unstructured Lagrangian data that is comprised of \( N \) pathlines. A schematic of a pathline from such a data set is presented in Figure 1. The pathlines are enumerated by the counter...
p, while the lengths of each pathline in terms of timesteps are stored in the $1 \times N$ vector $\vec{L}$, such that the length of the current pathline is given by $\vec{L}(p)$. At each timestep along a pathline, the position $\vec{x}$, velocity $\vec{U}$ and acceleration $\vec{a}$ are measured. The timesteps along each pathline are enumerated by the counter $i$. Thus, the position, velocity and acceleration along pathline $p$ at its $i$th step are denoted as $\vec{x}(p, i)$, $\vec{U}(p, i)$ and $\vec{a}(p, i)$. Components of these three vectors are denoted using lower-case symbols, while the dimension is indicated by the counter $j$. As such, position, velocity and acceleration can be represented as $\vec{x}(p, i) = [x_1(p, i), x_2(p, i), x_3(p, i)]$, $\vec{U}(p, i) = [u_1(p, i), u_2(p, i), u_3(p, i)]$ and $\vec{a}(p, i) = [a_1(p, i), a_2(p, i), a_3(p, i)]$. From this data, the velocity-gradient tensor $\nabla \vec{U}$ is to be calculated. In three-dimensional space this tensor is represented as:

$$\nabla \vec{U} = \begin{bmatrix}
\frac{\partial u_1(p, i)}{\partial x_1} & \frac{\partial u_2(p, i)}{\partial x_1} & \frac{\partial u_3(p, i)}{\partial x_1} \\
\frac{\partial u_1(p, i)}{\partial x_2} & \frac{\partial u_2(p, i)}{\partial x_2} & \frac{\partial u_3(p, i)}{\partial x_2} \\
\frac{\partial u_1(p, i)}{\partial x_3} & \frac{\partial u_2(p, i)}{\partial x_3} & \frac{\partial u_3(p, i)}{\partial x_3}
\end{bmatrix}. \tag{1}$$

In the proposed gradient-evaluation methods to follow, a “good” initial estimate of this tensor for values of $p$ and $i$ must be determined. There are a number of ways to do so as described in the Introduction section. The current study uses a simple sphere-of-influence method to achieve this initial estimate. The sphere size was determined such that approximately 12 neighbouring nodes existed within the sphere surrounding each pathline, while tracks whose neighbouring-node count fell beneath 6 were discarded. Linear interpolation was then applied to determine the necessary estimates of the gradient at all $\vec{x}(p, i)$. The following gradient-evaluation method uses this initial estimate as a starting point to evaluate its objective function, which is then minimized iteratively.

### 2.2 Minimization of Square-Residual Sum from Vorticity-Transport Equation

The following method for evaluating spatial gradients attempts to minimize the sum of square residuals from the vorticity-transport equation evaluated along the entire length of each pathline. The vorticity-transport equation is shown below:

$$\frac{D\vec{\omega}}{Dt} = (\vec{a} \cdot \nabla)\vec{U} + \nabla^2 \vec{\omega}. \tag{2}$$
At modest Reynolds numbers, the vorticity-diffusion term of this equation can be neglected. Thus, the substantial derivative and the vortex stretching term should be equal at all points along the pathline. However, due to the error in calculating the spatial derivative from the experimental data, these differences result in a residual for all dimensions \( j \), and at every value of \( p \) and \( i \). The residuals can be squared and summed, resulting in the single residual \( R^l(p, i) \) for a timestep along a pathline:

\[
R^l(p, i) = \sum_{j=1}^{3} \left( \frac{D\omega_j(p, i)}{Dt} - (\vec{\omega}(p, i) \cdot \nabla)u_j(p, i) \right)^2. 
\]

As \( R^l(p, i) \) must be zero at all points along each pathline, these residuals can be summed along an entire pathline from 1 to \( p \) to produce an objective function \( O \) for the entire pathline. This function is to be minimized.

\[
O = \sum_{i=1}^{L(p)} \sum_{j=1}^{3} \left( \frac{D\omega_j(p, i)}{Dt} - (\vec{\omega}(p, i) \cdot \nabla)u_j(p, i) \right)^2. 
\]

We wish to now find the set of \( \nabla \vec{U}(p, i) \) along the entire pathline the minimizes \( O \), given our initial guess. To do so, \( \nabla O \) is first determined with respect to all nine elements of \( \nabla \vec{U}(p, i) \), along the entire length of the pathline. Thus, \( \nabla O \) is comprised of \( 9 \times L(p) \) equations. \( \nabla O \), which represents a single component equation of \( \nabla O \) is presented in Eq. 5:

\[
\nabla O = \frac{\partial \left( \sum_{i=1}^{L(p)} R^l(p, i) \right)}{\partial \left( \frac{\partial u_j(p, i)}{\partial x_k} \right)}, \text{ where } i = 1 \ldots L(p), j, k = 1, 2, 3. 
\]

\( O \) can thus be minimized by method of steepest descent by adjusting a step size alpha between iterations that minimizes \( O \) given the current gradient:

\[
O(\nabla \vec{U}(p, i) + \alpha \nabla O). 
\]

The final estimates of \( \nabla \vec{U}(p, i) \) represent the evaluations of the velocity-gradient tensors along the entire pathline that have been corrected using information from other pathlines in its vicinity.
While the method of steepest descent is very simple and only locally minimizing, it was chosen for its ease of implementation, as it is sufficient to demonstrate the method proposed.

Figure 2: A schematic of the water-filled towing tank and image-acquisition system used by the current study to obtain 4D-PTV data. A NACA-0012 profile (A) was towed from rest through a measurement volume located 15c downstream (B). The measurement volume was illuminated by 40mJ-per-pulse, 527nm laser (C). The motions of 100µm-diameter silver-coated glass microspheres were acquired by four Fastcam SA-4 cameras at 1500Hz (D), from which 4D-PTV data of the wake was derived.

3. Apparatus: Towing-Tank Facility and Image-Acquisition System

To test the gradient-evaluation methods presented in Sec. 2, “Shake-The-Box” 4D-PTV data was collected along the suction side of a towed NACA-0012 profile in a water-filled towing tank at Queen’s University. A schematic of the apparatus is shown in Figure 2. The span and chord of the wing were $L = 1\, \text{m}$ and $c = 0.3\, \text{m}$, respectively, while the tip-gap spacing was kept less than 5% of chord, thereby ensuring a bulk two-dimensional flow. The profile was towed from rest with no initial angle of attack at a constant velocity of $0.33\, \text{m/s}$, corresponding to a chord-based Reynolds number of $Re_c = 10^5$. The profile then pitched rapidly to an angle of attack of $\alpha = 35^\circ$ in order to form a strong dynamic stall vortex, such that physically sensible velocity-gradient estimates could be quickly identified from the large coherent structure. The profile’s suction side passed through a $0.5c \times 0.5c \times 0.1c$ measurement volume arranged such that the normal-vector of the laser plane was parallel with the span of the wing. The volume was located
15c downstream from the start position, thereby ensuring that the profile was in quasi-steady conditions prior to the onset of motion. The measurement volume was achieved via a 40mj-per-pulse, 527nm laser, which illuminated 100μm-diameter silver-coated glass microspheres. The motions of the microspheres were acquired by four Photron Fastcam SA-4 cameras sampling at 1500Hz. The Stokes number of the particles was approximately $3 \times 10^{-3} \ll 0.1$, which ensured tracer-accuracy errors of < 1% [11]. To minimize image distortions, water filled prisms were fixed onto the glass pane of the tank such that all cameras were orthogonal to a prism face. The images were then imported and processed in DaVis 8.2.0 (LaVision GmbH, Goettingen, Germany) for 4DPTV analysis. Lagrangian data were subsequently exported to MATLAB 2012a (Mathworks, Natick, MA, USA) for post-processing. An initial estimate for the elements of the velocity-gradient tensor along the entire lengths of all pathlines was calculated using the sphere-of-influence method described in Sec. 2.1. As a proof-of-principle study, the gradient-evaluation method based on the vorticity transport equation was applied to all pathlines. The proceeding section shows preliminary results achieved via this method. The second gradient-evaluation method that utilizes the flow-map gradient is left as future work.

4. Results & Discussion

An indicative flow-field snapshot prior to correction is shown in Figure 3, with the particle field colored by spanwise vorticity. The dynamic stall vortex is shown clearly, and the particle tracks in this region are long-lived, providing excellent temporal information for the physical constraint methodology outlined above. A typical pathline optimization is also shown in Figure 3. As a consequence of the gradient-descent method, only a local minimum of the objective function from Equation (4) was obtained. However, this local minimum was obtained in under ten iterations, allowing the entire flow-field to be optimized within a few hours on a typical desktop PC.

By enforcing the dynamic constraint of the vorticity equation, the spatial gradients observed in the flow field exhibit much greater coherence in both space and time. Specific flow patterns that could not be previously identified emerge, such as that exemplified by the flow field shown in Figure 4. Specifically, in the vorticity field, which is on the left of Figure 4, a region of low-vorticity fluid emerges. The region is highlighted by the red contour. This low-vorticity region coalesces with the high-vorticity dynamic stall vortex core, highlighted by the blue contour. The vortex core exhibits vortex compression, corresponding to a reduction of vorticity magnitude. Meanwhile, the low-vorticity region is associated with vortex stretching, corresponding to an increase in vorticity
magnitude. This is consistent with the two regions coalescing, as observed in the vorticity field. Such details are impossible to observe if the vortex stretching field is computed using the initial velocity gradients, indicating that the gradient-correction scheme dramatically improves the signal-to-noise ratio without smoothing the data such that it loses these small-scale features.

Figure 3: A snapshot of the flow-field shows the development of a strong coherent dynamic stall vortex (left). Considering an individual pathline from this data set, the optimization scheme outlined above converges rapidly, on the order of 10 iterations (right).

Figure 4: The final constrained gradients allow the identification of flow patterns not previously resolvable. Here, a region of low-vorticity magnitude, highlighted by red contours, coalesces with a region of high-vorticity magnitude, highlighted by a blue contours.

5. Conclusions and Outlook

We have presented a method for correcting noisy velocity-gradient data by enforcing known physical constraints, and by exploiting the long time history of individual particles in 4D-PTV. Moreover, we have presented a basic implementation of the method to demonstrate its robustness and low computational cost. This method produces the effect of a discriminating filter, reducing noise levels while preserving the small-scale flow structures.

However, the validation of this method is incomplete. Immediate future work will include a comparison to traditional spatial and temporal filtering, as well as the development of a dedicated
validation test case. Moreover, the current optimization technique uses a dynamic equation of motion. As the principle advantage of Lagrangian data is direct access to the flow map, in addition to high-quality temporal derivatives, a kinematic equation of motion that utilizes the flow map directly is an attractive alternative currently under development.

References