On the suppression of PIV measurement noise with a POD based filter

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Abstract Random noise removal from Particle Image Velocimetry data and spectra is of paramount importance, especially for the computation of derivative quantities. Data filtering is critical, as a trade-off between filter effectiveness and spatial resolution penalty should be found. In this paper a filtering method based on Proper Orthogonal Decomposition and low order reconstruction is proposed. A criterion to perform the choice of the optimal number of modes based on the minimization of both the reconstruction error and the signal withdrawal is proposed too. The method is validated via synthetic and real experiments, showing a substantial reduction of the measurement uncertainty. As prototype problems we consider PIV vector fields obtained from channel flow DNS data and from an experiment on fractal generated turbulence. We determine the optimal number of modes to be used for the low order reconstruction in order to minimize the measurement error. The spatial resolution in the spectra is doubled as well as the capability of detection of coherent structures is improved. Distortions of the measured turbulent dissipation distributions due to amplified noise are consistently reduced, thus improving the capability of PIV as an instrument for turbulent flows investigation.

1. Introduction

The key of the success of Particle Image Velocimetry (PIV) lies in its ability to measure the instantaneous velocity field simultaneously at many points with spatial resolution sufficient to allow the computation of the instantaneous fluid vorticity and rate of strain. To date, PIV is the only experimental method that provides snapshots of the two or three-components velocity vector fields on a planar or volumetric cross section of rapidly evolving flows (Westerweel et al. 2013). Unfortunately experimental noise and spuriously deleted vectors in PIV data pose great challenges to the reliability of velocity derivatives and of the gradient based quantities. For instance, for tomographic PIV measurements, the standard deviation of the divergence computed on raw data of an incompressible flow can be assumed as an estimation of the accuracy of measurement of the velocity spatial derivatives and it is typically found to be around 7% (Ceglia et al. 2014) of the maximum vorticity in the measured flow field (locally the error can be much higher). Even using advanced PIV algorithms and temporal filtering of data, error of 3% in the vorticity measurements are reported (see, e.g. Violato et al. 2012).

While tomographic PIV measurements present also an error component related to tomographic reconstruction, which cause both a gradient modulation and random noise (Elsinga et al. 2011, Discetti et al. 2013), PIV measurement errors due to correlation have been deeply studied and can be distinguished in mean-bias error and root-mean-square (RMS) error of the remaining fluctuations. The bias error is typically appears in the form of peak-locking, responsible for measurement errors up to 0.1 pixels (Westerweel1997) or as a modulation error that arises from the spatial filtering effects of interrogation window (Scarano 2003). Even though the bias error received a more significant attention from the community of PIV developers, the RMS error is often the dominant component of the measurement uncertainty. According to Adrian (1991) the RMS is proportional to the particle image diameter (and hence to the correlation peak width). Other sources of RMS errors are the change of relative intensity between particle images between exposures due to out-of-plane motion, fluctuating background intensity and camera noise introduced during the recording process. Westerweel (2000) reported a typical figure of 0.05 pixels for the RMS error. The RMS error is also reported to be highly sensitive to the interrogation procedure: the use of window weighting functions and of advanced interpolators is also shown to affect the amplitude of the random error (Astarita2006, 2007).

Moreover, false correlation peaks detection mostly occurs when the correlating windows produce an insufficient number of particle image pairs, resulting in the occurrence of spurious vectors (Huang et al. 1997). Recognizing and eliminating such incorrect vectors is a mandatory step to obtain undistorted velocity
statistics. This procedure is referred to as data validation (Westerweel 1994, Westerweel and Scarano 2005). One path to reduce the instantaneous measurement uncertainty consists in using temporal information from time-resolved images or from multiple exposures to reduce the measurement errors. Nowadays the availability of high speed lasers has multiplied the number of attempts in this direction (see the recent works by Cierpka et al. 2012, Sciacchitano et al. 2012, Lynch and Scarano 2013).

In case of non time-resolved data the chances to reduce the measurement uncertainty are very limited. In this work we propose to exploit the availability of large ensembles of velocity fields as a statistical filter. Proper orthogonal decomposition (POD, Sirovich 1987) allows for a decomposition of the field in its principal components. By taking into account just the first modes it is possible to extract the most important structures in the field. As a matter of fact, POD acts as a filter on the data and at the same time throws into the single snapshots information from the entire ensemble.

The reconstruction of the flow field with a limited set of modes has been used mainly for two scopes: the filtering of data to individuate the different scales of coherent structures and the replacement of outlier vectors from PIV images. In particular Adrian et al. (2000) suggested to use the decomposition of the velocity field to filter the data and individuate the vortices. Karri et al. (2009) firstly used the POD to filter the gradient estimate from a PIV velocity field. Everson and Sirovich (1995) first proposed to use POD to reconstruct gappy data. Recently, Raben et al. (2012) applied this concept to the outliers replacement, and introduced an adaptive criterion for the number of modes selection, based on field smoothness evaluated concurrently to the field reconstruction.

The present work will show that it exists an optimum number of modes to reconstruct the flow field while minimizing the measurement noise. Furthermore it will propose a criterion for the selection of this optimum number of POD modes.

2. Methodology

The Proper Orthogonal Decomposition (POD) is a mathematical procedure that identifies an orthonormal basis using functions estimated as solutions of the integral eigenvalue problem known as Fredholm equation (see Fahl 2000 for a rigorous formulation). Consider a data matrix, that for the case of PIV is the sample ensemble of the fluctuating part of the velocity field according to the Reynolds decomposition, \( U \in \mathbb{R}^{n \times p} \), where \( n \) is the number of samples and \( p \) is the number of grid points. \( U \) can be decomposed as:

\[
U = \Psi \Sigma \Phi
\]  

where \( \Phi \) and \( \Psi \) constitute the decomposition basis of the fluctuating velocity field \( U \), respectively in space and time, and \( \Sigma \) is a diagonal matrix containing the singular values associated to the fluctuating field. The solution is not unique as it depends on the chosen basis functions. The snapshots method proposed by Sirovich (1987) assumes that the POD modes are calculated as the eigenmodes of the two-point temporal correlation matrix. When computed with this method, the mode number that contributes to the field equals the one of the snapshot used.

As the eigenmodes are ordered by their energy contribution, the field reconstructed with just the \( k \) most relevant modes is given simply by:

\[
U_k = \Psi \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix} \Sigma \Phi
\]  

where \( U_k \) is the reconstructed flow field, \( I_k \) indicates the rank \( k \) identity matrix and \( 0 \) indicates that the matrix containing \( I_k \) is a square matrix of dimension \( n \times n \) with zero entries except for \( I_k \).

A qualitative argument on the existence of an optimum value of \( k \) for the suppression of noise from a set of PIV images could be easily derived if we consider the measured field \( \tilde{U} \) as a sum of objective fluctuating field \( U \) (that is the measured field depurated from the random error) and measurement random error \( \varepsilon \). It should be:
\[ \tilde{U}_k = U_k + E_k \approx \Psi \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}(\Sigma + \Sigma_e) \Phi \]  

(3)

where \( \Sigma_e = \Phi^T E \Psi^T \). In this case the true flow field should have an energy distribution that monotonically decreases and asymptotically goes to zero, as the bulk of energy is mostly concentrated on coherent structures that are bigger than the measurement spatial resolution (Berkooz et al., 1993). The measurement random error (that is spatially uncorrelated as we do not consider the modulation due to finite spatial resolution) should have a nearly uniform energy distribution all over the POD modes. In practice, it is expected to have the spectral behavior of white noise. As a consequence an optimum number of modes \( k^* \) can be identified beyond which the energy content of noise is comparable or higher than the one of the real signal. Dropping all the following modes after this \( k^* \) will give for sure that the energy of the real flow discarded is lower than the one of the discarded noise.

We define the reconstruction error \( \delta_{RT} \) with respect to the true flow field as:

\[ \delta_{RT} = \left( \frac{1}{np} \text{Tr} \left( (U - U_k)(U - U_k) \right) \right)^{0.5} \]  

(4)

where \( U_k \) is the velocity of the reconstructed fluctuating field. It can be demonstrated that \( \delta_{RT} \) defined in Eq. (4) has a minimum in \( k^* \) (for a mathematical treatment please refer to Raiola et al. 2014). It can be also demonstrated that the order of magnitude of \( k^* \) can be calculated from Eq. 5:

\[ k^* \approx \left( \frac{n q}{\sigma^2 \zeta(5/3)} \right)^{3/5} \]  

(5)

where \( q \) is the average turbulent kinetic energy of the true field, \( \sigma \) is the standard deviation of the random error and \( \zeta(s) \) is the Euler-Riemann zeta function of \( s \). Considering a white spectrum for the random error, the number of snapshots \( n \) to be used in the decomposition can be estimated with Eq. 6:

\[ n \approx (1 - RF)^{-5/2} \left( \frac{q}{\sigma^2 \zeta(5/3)} \right)^{2/5} \]  

(6)

where \( RF \) noise reduction factor (i.e. the residual error energy in the reconstructed fields is \((1 - RF)\sigma^2\)). Beside Eq. 5 can be an useful formula to predict the order of magnitude of the optimal number of modes, it can not be considered an operative criterion for the choice of \( k^* \) as typically knowledge during PIV experiments is limited to the measured velocity fields \( \bar{U} \) and no information is available on the objective field and the random error. An alternative criterion to find the optimal number of modes to use in the reconstruction is then required. The flow field measured with the correlation of PIV images (referred from now on as “measured”) is analyzed with POD obtaining the principal components. The reconstruction of the snapshots is performed at different values of the number of modes. These reconstructed fields (referred in the following as “reconstructed”) are compared with the “measured” fields. We define the reconstruction error \( \delta_{RM} \) with respect to the measured field as:

\[ \delta_{RM} = \left( \frac{1}{np} \text{Tr} \left( (\bar{U} - U_k)(\bar{U} - U_k) \right) \right)^{0.5} \]  

(7)

where \( \bar{U}_{\text{meas}} \) is the velocity of the “measured” field. \( \delta_{RT} \) is a monotonically decreasing function of the number of modes used for the reconstruction and is equal to 0 for \( k = n \). It can be demonstrated that the function \( F(k) \), defined as:

\[ F(k) = \frac{\delta_{RM}^2(k + 1) - \delta_{RM}^2(k)}{\delta_{RM}^2(k) - \delta_{RM}^2(k)} \]  

(8)
may reach asymptotically 1, for $k$ sufficiently large, up to the point in which the POD eigenvalues differences are comparable to the random noise. In the next section it will be shown that a reasonable threshold for the number of modes to retain can be set at $F(k^*) = 0.999$. This is formally equivalent to look for an elbow in $\delta_{RM}$, and can be considered a restatement of the classic scree test plot (Cattell 1966) used in PCA to determine the number of components to retain in a low order reconstruction.

### 3. Numerical validation on synthetic images

A synthetic experiment is performed with the DNS database of channel flow at Reynolds number equal to 4000 (based on the bulk velocity $U_b$ and twice the half-channel height $h$) from John Hopkins Turbulence Database (Li et al. 2008; Yu et al. 2012; Graham et al. 2013). The DNS domain size is $8\pi h \times 2h \times 3\pi h$. The data are stored in a $2048 \times 512 \times 1536$ points grid. For the purpose of the PIV synthetic experiment bi-dimensional square sub-domains with one side corresponding to the wall, covering regions with size $h \times h$ and parallel to $x$-$y$ plane are extracted from DNS domain. These sub-domains are used to generate synthetic images with dimensions $1024 \times 1024$ pixels (resulting in a resolution of 4 pixels/grid DNS points).

The displacements are multiplied by a scaling factor in order to achieve a mean displacement on the channel centerline equal to 15 pixels. Gaussian particles (mean diameter 3 pixels, standard deviation 0.5 pixels, 300 counts peak intensity) are randomly generated with a particle density of 0.01 particles per pixel. The laser intensity is simulated to be Gaussian (with half power width equal to 4 pixels along the thickness of the light sheet) in order to take into account the effect of correlation degradation due to the out-of-plane motion.

Noise with uniform distribution (maximum intensity 50 counts, standard deviation 14.4 counts) is added on the images. The interrogation strategy is an iterative multi-step image deformation algorithm, with final interrogation windows of 32x32 pixels, 75% overlap. A Blackman weighting window is used to improve the stability and the spatial resolution (Astarita 2007).

The spatial resolution achieved in this simulated experiment is realistic and consistent with that of recent PIV experiments (see, e.g. Hong et al. 2012 achieving a resolution of about $125 \times 60$ vectors in a $2h \times h$ domain).

The standard deviation of the random error $\sigma$ is estimated by interrogating images with zero-imposed displacement and same background noise feature, and it is found to be equal to 0.18. The mean turbulent kinetic energy $q$ in the ensemble is of about 1.29 square pixels.

![Graph](image.png)

**Fig.1** $\delta_{RT}$, $\delta_{RM}$ (left axis) and $F$ (right axis) versus the number of modes used in the reconstruction.
Fig. 2 Instantaneous fluctuating vorticity ($\omega_x$) field: a) the DNS field used for this benchmark; magnified view of the DNS field (b), measured field (c), field reconstructed with 1200 modes (d), field reconstructed with 3000 modes (e).
According to Eq. 6, for a random error reduction of 80%, a number of 8000 synthetic images is chosen to form the ensemble. With the same values, the expected optimal number of modes from Eq. 5 is of about 1300 modes.

The 8000 images are obtained by using flow fields with a time separation equal to 250 DNS timesteps (corresponding, in our resolution, to a peak displacement of 250 pixels) and with a space separation along the z direction equal to 0.23 h (235 pixels). The flow fields are taken at two streamwise positions, i.e. \( x = 0 \) and \( x = 4 \Delta h \). The number of images is doubled by flipping data from the other half of the channel. \( \delta_{RT}, \delta_{RM} \) (left axis) and F (right axis) versus the number of modes used in the reconstruction. The values of \( \delta_{RT}(k) \) and \( \delta_{RM}(k) \) as defined by Eq. 4 and Eq. 7 are plotted; data are presented in non-dimensional form dividing by \( \delta_{MT} \), i.e. the error of the measured field with respect to the true one, that in this test is equal to 0.32 pixels. It has to be outlined here that this value is higher than the random noise since it includes also signal modulation. The minimum \( \delta_{RT} \) is reached when the first 1200 modes are used for the reconstruction (corresponding to 96.6% of the fluctuating energy of the measured field).

It has to be remarked that a quite extended plateau of about 500 modes width is present, thus the exact choice of the number of modes to be retained is not critical, provided that one can identify a reasonable estimate of the range in which the optimum should be. In this plateau the error is of about 0.82 \( \delta_{MT} \) (corresponding to a 0.06 pixels total error reduction). Beyond 1200 modes \( \delta_{RM} \) has approximately a linear trend with \( k \). Indeed, as noise has a white spectral distribution, it is reasonable to expect that for large \( k \) the contribution of each mode is approximately constant.

The function \( F(k) \) defined in Eq. 8 is also plotted; its values are low-pass filtered. The function \( F(k) \) approaches one beyond 1200 modes, apart from the very last coefficients in which the linearity of the contribution of noise is lost. A reasonable threshold is \( F(k) = 0.999 \), corresponding to \( k \approx 1300 \) and to a \( \delta_{RT} \) nearby the minimum.

In Fig. 2 a true snapshot is compared to the raw snapshots obtained with PIV interrogation and with POD-based low order reconstructions. Maps of the out-of-plane vorticity component are reported to stress differences. The measured field (Fig.2c) is affected by spurious vortical features that are not present in the original field. As an example, the negative vorticity peak marked as A in the figure is much weaker in the original field, while in the measured field has intensity comparable to that of the vortex marked as B, that is an original field feature. The optimal reconstruction with 1200 modes (Fig.2d) provides an astonishing improvement of the data quality with respect to the measured field. This reconstruction smears out the negative vorticity peak in A, while retaining vortex B. Such a result could not be achieved by a spatial filter since size and intensity of the vorticity peaks in A and B in the measured field are very similar. Even though the residual error \( \delta_{RT}(k^*) \) from Fig. 1 might appear still relatively large, it has to be reminded that in this simulated experiment the measurement error is dominated by bias due to finite spatial resolution.

Going very far from the optimal number of modes (e.g. 3000 modes in Fig.2e) determines a stronger noise contamination, and, for instance, the vortex A is still present.
In Fig. 3 the power spectra of the true, the measured and the reconstructed fields are reported. The wavelengths are expressed in pixels. Data are proposed in the form of the compensated spectrum (i.e. multiplied by the wavenumber to the 5/3) in order to magnify the effects at the smallest scales. It has to be remarked that, even in the case of the original DNS data, the spectrum at the small scales is contaminated by aliasing effects due to the finite length of the domain. The reconstruction with 1200 modes closely follows the DNS spectrum for a wider portion of the wavelengths with respect to reconstructions with a larger number of modes. The reconstruction with optimal number of modes allows for a high fidelity prediction of the true spectrum up to a wavelength of 70 pixels. The growth of the error with respect to the true spectrum for smaller wavelengths can be associated mostly to the residual noise. For the sake of completeness, the spectrum obtained by a reconstruction with a lower number of modes is also reported. The reconstruction with 300 modes (which corresponds to 95% of the fluctuating energy) causes a significant underestimation of the spectral energy of a wide range of large scales, thus highlighting that the information obtained from these modes is still insufficient to achieve a proper description of the flow field.

3. Experimental validation on free jet with fractal grid turbulator

The POD-based filter is tested on an experimental PIV dataset. The flow field under investigation is the outflow of a circular free jet with a fractal grid turbulator (Cafiero et al. 2014b; for more details on fractal grid turbulence refer to Valente & Vassilicos 2011; Gomes-Fernandes et al. 2012).

![Diagram](image)

**Fig. 4** A sketch of the fractal grid used in the experiment (courtesy of Cafiero et al. 2014b).

![Graph](image)

**Fig. 5** $\delta_{RM}$ (left axis) and $F$ (right axis) versus the number of modes used in the reconstruction.
The jet blows from a short-pipe nozzle (diameter $D = 20\, mm$, length $6.2\, D$, with a smooth contraction at its inlet) installed on the bottom of a nine-sided water tank facility (internal diameter $600\, mm$, height $700\, mm$). The bulk mean velocity $U_b$ of the fluid entering the chamber is about $1.4\, m/s$, thus resulting in $Re_p = U_b\, D/v = 28000$ where $v \approx 10^{-6}\, m^2/s$ is the kinematic viscosity of water.

The experiment is performed with the fractal grid sketched in Fig.4 mounted at the nozzle exit section. The grid is laser-cut from a $0.5\, mm$ thick aluminum foil. The fractal shape is obtained by repeating a square pattern at three different scales, halving at each iteration the length and the thickness. At the first iteration length $L_0$ is $10\, mm$ and thickness $t_0$ is $1\, mm$. The blockage ratio of the grid is equal to 0.32.

The PIV setup is similar to that of Cafiero et al. (2014b). The flow is seeded with neutrally buoyant polyamide particles with an average diameter of $56\, \mu m$. The light source is a Laser Gemini PIV ND:Yag ($200\, mj/pulse$). Acquisition is performed with a PCO Sensicam ($1280 \times 1024$ pixels resolution, with $6.7$ micron pixels pitch) equipped with a $50\, mm$ focal length objective. The average magnification is about 0.16, resulting in a resolution of approximately $480\, pix/D$ ($24\, pix/mm$). The PIV interrogation strategy, applied on cropped images with size $800 \times 700$ pixels, is an iterative multi-step image deformation algorithm, with final interrogation windows of $24 \times 24$ pixels, $75\%$ overlap. The size of the ensemble used to compute the POD modes is set to 11000 images.

This test case is particularly significative since the generation of turbulence simultaneously at multiple scales poses a severe challenge to anemometric measurement techniques in terms of dynamic range (see, for instance, the assessment of PIV performances conducted by Discetti et al. 2013 on fractal generated turbulence).

![Instantaneous fluctuating vorticity ($\omega_x$) field from PIV experiment: a) the measured flow field; magnified view of measured fluctuating field (b), fluctuating field reconstructed with 1800 modes (c), reconstructed with 5000 modes (d).](image-url)
The values of $\delta_{BM}(k)$ and of the function $F(k)$ (low-pass filtered to remove the effect of the noise) are plotted in Fig. 5. Compared with the previous synthetic benchmark, a similar shape in the error with respect to the measured flow field $\delta_{BM}(k)$ is found. By taking as a criterion a value of $F(k^*) = 0.999$, an optimum number of modes of nearly 1800 is found.

In Fig. 6a magnified view of a raw snapshot obtained from PIV interrogation (Fig. 6b) is compared with snapshots reconstructed with 1800 modes (Fig. 6c), corresponding to 94.8% of fluctuating energy, and with 5000 modes (Fig. 6d), corresponding to 98.7% of fluctuating energy. The regions indicated with the letters A, B, C, D and E highlight region of strong distortion of the flow pattern which are very likely to be related to clusters of outliers undetected by the validation algorithm. The optimal reconstruction corrects the distorted vectors, restoring the continuity of the flow field. Moreover a general increase in data smoothness and a reduction of background noise is observed. A larger number of modes (Fig. 6d) deals to the persistence of flow field distortion and reduced capability of suppressing regions of spurious vectors.

From the transverse velocity spectra calculated on the jet axis (Fig. 7) it is evident that the reconstruction with 1800 modes smears out part of the energy content in the high frequencies (which is more likely to be related with noise) while still retaining the energy content in the larger scales as obtained from the direct measurement. In agreement with the observation from the synthetic test, using a lower number of modes leads to a significant underestimation of the turbulent energy also at the large scales, thus, as a matter of fact, resulting in a sacrifice in terms of spatial resolution.
In Fig. 8 the profiles of the dissipation over a jet transversal section at \( y/D = 1.15 \) of the raw dataset and of the dataset reconstructed with the optimal number of modes are compared. The dissipation is calculated via the Eq. 9 (derivatives are computed with a second-order central difference scheme):

\[
\varepsilon = 2\nu < S_{ij}S_{ij} >
\]

While at this distance the measured dataset reveals only the presence of the peak associated with the shear layer of the jet, the reconstructed dataset predicts the persistence of the effects of the wake developing downstream of the largest square of the fractal grid.

5. Conclusions

A filter for randomly perturbed flow fields (such as PIV data) based on random error minimization in the low-order reconstruction with an optimal number of POD modes is presented. An empirical criterion for the choice of the optimal number of modes to retain is also indicated. The sensitivity to this choice is quite limited. Furthermore, the optimal number of modes is obtained by simply observing the variance of the difference between the reconstructed fields and the original ones, without adding any hypothesis on the features of the flow field itself, thus making the method very robust and flexible. The method is validated on a synthetically generated ensemble generated from channel flow DNS data set. In this test case the error is dominated by modulation effects, which cannot be recovered by the POD low order reconstruction. Nevertheless, a total error reduction of about 18% is achieved, corresponding to a value that is about the 33% of the noise estimated with the zero-displacement benchmark test, while retaining most of the spectral informations of the original field. The reduction of the random error is extremely beneficial in the computation of derivative quantities, which are more prone to be affected by noise amplification. The vorticity distributions highlight that the method is able to suppress spurious vorticity blobs without removing original vortical structures with comparable size and intensity. The spectra of the data closely follow those of the DNS for a wider portion of the wavelengths with respect to reconstruction with a larger number of modes (up to a wavelength slightly larger than twice the interrogation window size). As expected, the improvement is more remarkable on derivative quantities (see, for instance, the vorticity spectra). Finally the method is tested on a real PIV measurement of the near field of a circular jet with a fractal turbulator. Similarly to the synthetic test, the improvement on the derivative quantities computation is sensitive (see, for instance, unphysical features of the turbulent dissipation map). Furthermore, the low-order reconstruction is very effective in removing clusters of spurious vectors, which generally pose a challenge to the standard validation methods.

The proposed method is expected to contribute in enhancing the reliability of PIV as it allows for uncertainty reduction and robustness improvement. The spatial resolution might also indirectly benefit from it. Indeed, generally the choice of the processing algorithm (window size, interrogation method, etc.) arises from a trade-off between desired spatial resolution and measurement noise amplification. The optimized low-order reconstruction allows for the use of advanced high-resolution interrogation algorithm since the random noise can be consistently reduced in post-processing. As turbulent flows investigation is demanding in terms of dynamic range requirements, this advancement would be beneficial.

Acknowledgments

The authors wish to thank Gioacchino Cafiero for contributing the realization of the jet experiment.

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