Longitudinal Cross-correlation Function in Fully Turbulent Flow

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Abstract The application of a simple statistical model to transform temporal correlation functions from one-point measurements into two-point longitudinal spatial cross-correlation functions is investigated. Instead of Taylor’s frozen-flow hypothesis a simple advection model with fluctuating velocities and their probability density function is used. This model is valid for long Lagrangian correlation. Based on two-point laser Doppler measurements taken in a turbulent round free air jet, it is shown that it correctly reproduces the time shift including the observed lagging of the correlation peak compared to the mean velocity, the decaying correlation strength, the blur and the arising skewness. This transform is extended to a more flexible advection model, including the decrease of the Lagrangian correlation during the passage of fluid portions through the arrangement of measurement volumes.

1. Introduction

While a direct derivation of the spatial longitudinal cross-correlation function $\rho(\xi)$ in a turbulent flow requires two-point measurements, Eulerian temporal autocorrelation functions $\rho_h(\tau)$ can be derived from velocity measured at one point as a function of time. In turbulent flows with significantly larger mean velocity $U$ compared to the fluctuating velocity $u'$, usually Taylor’s frozen flow hypothesis (Taylor, 1938) (TH) is widely used as means of transforming temporal autocorrelation functions derived from one-point measurements into spatial longitudinal cross-correlation functions. The advection velocity is assumed to be constant, namely the mean velocity. The correspondence between the time lag $\tau$ and the longitudinal separation $\xi$ in the case of Taylor’s frozen flow hypothesis is given by

$$\xi = U\tau.$$

2. Time-space correlation for small decay of the Lagrangian correlation

Nobach and Tropea (2012) introduced an integral time-length transformation (ITLT), which allows fluctuations of the advection velocity. Assuming a random advection velocity $u$ with a certain probability density function $f(u)$. The velocity is assumed to be constant over the time of flight within the relevant separations of the cross-correlation function. This corresponds to the Lagrangian temporal correlation function, which must not much decay within the time scales of time of flight between the measurement volumes. Above this time scale the velocity may change yielding a random value for each instance in time. The transformation from the Eulerian temporal autocorrelation function into the spatial longitudinal cross-correlation function then reads

$$\rho_h(\xi) = \int \rho_h(-\theta) f_u(\theta | \xi) d\theta,$$

where $f_u(\theta | \xi)$ is the probability density function of the time of flight $\theta$ for a fluid element to travel the distance $\xi$, which depends on the probability density $f(u)$, yielding

$$f_u(\theta | \xi) = \frac{\xi}{\theta} f\left(\frac{\xi}{\theta}\right).$$
Introducing an additional time lag $\tau$, temporal cross-correlation functions $\rho(\tau, \xi)$ can be derived for any longitudinal separation $\xi$ between two points in the flow (time-space correlations) as

$$
\rho(\tau, \xi) = \int_{-\infty}^{\infty} \rho_{\xi}(\tau - \theta) f_{\xi}(\theta | \xi) d\theta.
$$

The spatial longitudinal cross-correlation function and the Eulerian temporal correlation function correspond to the time-space correlation via

$$
\rho_{\xi}(\xi) = \rho(0, \xi)
$$

and

$$
\rho_{\tau}(\tau) = \rho(\tau, 0).
$$

3. Measurement in a turbulent round free air jet for small decay of the Lagrangian correlation

On the centerline of a turbulent round free air jet measurements of the stream-wise velocity component have been done with a two-point laser Doppler velocimeter, 40 inner diameters downstream the nozzle (Fig. 1). The separation has been varied between 0 and 4 inner diameters of the nozzle. Specifications of the flow field are given in Table 1 and those of the laser Doppler system in Table 2. From the two-point measurements, temporal cross-correlation functions (CCF) have been calculated as reference. From the measurements with no separation one long data set has been recorded and the Eulerian temporal autocorrelation function (ACF) has been calculated. From the ACF two predictions of the time-space correlation function have been derived, one using Taylor’s frozen flow hypothesis (TH) and one using the integral time-length transform (ITLT).

Fig. 2 shows the two predictions in comparison. While TH simply shifts the autocorrelation peak in time according to the assumed separation $\xi$ and the mean velocity $U$, which clearly disagrees with the measured two-point correlations as well as with the expected behavior of a turbulent structure advected over time and space, ITLT shows a much more complex behavior, which nicely agrees with the measured two-point correlations, reproducing besides the blur and the decrease of the correlation strength, also the time shift including the observed lagging of the correlation peak compared to the mean velocity and a skewness of the correlation peak correctly.

Even if redundant with the correct reproduction of the time-space correlation, Fig. 3 shows the first 15 complex Fourier coefficients of the transfer function for the same case as in Fig. 2. While TH rotates in the complex plane with constant magnitude, ITLT reproduces the decreasing amplitude with increasing frequency and the appropriate phase correctly.

4. Time-space correlation for finite Lagrangian correlation

The transformation from the Eulerian temporal autocorrelation function into the time-space correlation

$$
\rho(\tau, \xi) = \int \rho_{\tau}(\theta) \rho_{\xi}(\tau - \theta) f_{\xi}(\theta | \xi) d\theta
$$

now introduces the Lagrangian correlation $\rho_{\tau}(\theta)$ and requires a probability density function $f_{\xi}(\theta | \xi)$ of times of flight between measurement volumes having the distance $\xi$ considering both, the random velocity $u$ with the probability density function $f(u)$ and the decaying Lagrangian correlation $\rho_{\tau}(\tau)$ and this for multiple arrivals (for strong turbulence degrees a single fluid element may arrive at the distance $\xi$ multiple times).

To derive the probability density $f_{\xi}(\theta | \xi)$, first the probability density $f_{\xi}(\xi | \theta)$ of distances $\xi$ reached after time $\theta$ is investigated.
Assuming a Lagrangian correlation \( \rho_1(\tau) \), decaying significantly on the scales of the time of flight between the measurement volumes, the distance \( \xi \) traveled in time \( \theta \) becomes random with a mean \( \mu_\xi \) and a variance \( \sigma^2_\xi \), both as functions of \( \theta \). Modeled as a dispersion process one gets

\[
\mu_\xi(\theta) = U\theta
\]

and from Taylor (1921)

\[
\sigma^2_\xi(\theta) = 2\sigma^2 \int_0^\theta \rho_1(\eta) d\eta d\tau.
\]

which is, following Wilson and Zhuang (1989)

\[
\sigma^2_\xi(\theta) = 2\sigma^2 \int_0^\theta (\theta - \eta) \rho_1(\eta) d\eta,
\]

where again, \( U \) is the mean velocity and \( \sigma^2_\xi \) is the variance of the velocity.

Unfortunately, in the case of correlation of the velocity over time, the velocity at time \( \theta \) is not independent of the distance traveled during this time. Therefore, it is necessary to consider different velocities \( u \) and find the probability density function \( f_\xi(\xi|\theta,u) \), which is the conditional probability density function of locations \( \xi \) at time \( \theta \) under the condition that the velocity at time \( \theta \) is \( u \), which reads

\[
f_\xi(\xi|\theta,u) = \frac{1}{\sqrt{2\pi\sigma^2_\xi(\theta|u)}} e^{-\frac{[\xi - \mu_\xi(\theta|u)]^2}{2\sigma^2_\xi(\theta|u)}}
\]

with

\[
\mu_\xi(\theta|u) = \mu(U + (u - U) \int_0^\theta \rho_1(\eta) d\eta)
\]

and

\[
\sigma^2_\xi(\theta|u) = 2\sigma^2 \int_0^\theta (\theta - \eta) \rho_1(\eta) d\eta - \sigma^2 \left[ \int_0^\theta \rho_1(\eta) d\eta \right]^2.
\]

For a given \( u(\theta) \) from a random distribution with the probability density function \( f(u) \) the probability density function of times of flights \( f_\theta(\theta|\xi) \) is proportional to the probability density of distances \( f_\xi(\xi|\theta,u) \), that a fluid element has reached the distance \( \xi \) at time \( \theta \) under the condition of the velocity \( u \) at time \( \theta \)

\[
f_\theta(\theta|\xi) \propto f_\xi(\xi|\theta,u)
\]

and to the magnitude of the velocity \( u \)

\[
f_\theta(\theta|\xi) \propto |u(\theta)|.
\]

Integrating over all possible velocities \( u \) then yields

\[
f_\theta(\theta|\xi) = \frac{c}{u} \int f_\xi(\xi|\theta,u) f(u) du
\]
where \( c \) must be set such that

\[
\int_0^{\infty} f_{\delta}(\theta | \bar{x}) d\theta = 1.
\]

All these integrals can be derived numerically from the measured data. The Lagrangian correlation function, required also for the transform, unfortunately, cannot be derived directly from the measured data. A rough estimate can be obtained from the integral time scales from the two-point cross-correlation functions at different distances of the measurement volumes. The primary function, therefore, is a function of the distance of the measurement volumes, which must be transformed into a function of time. This can be done in various ways, as discussed in the present paper. Since the estimation of the Lagrangian correlation is not very critical, a rough estimate is enough here. Since the TH is good enough for this purpose, it has been used here to keep things as simple as possible. The whole procedure then yields a new prediction of the correlation function under the condition of decaying or finite Lagrangian temporal correlation.

5. Measurement in a turbulent round free air jet for finite Lagrangian correlation

To test the prediction for finite Lagrangian correlation, a second experiment has been performed with a different flow, which has a significant decay of the Lagrangian correlation within the time scale of the times of flight between the measurement volumes. The measurements have been taken again on the centerline of a round turbulent air jet in a distance of 40 inner diameters with separation of the measurement volume between 0 and 10 inner diameters. The respective flow specifications are given in Table 3.

The calculations are similar to the previous case, however, additionally the prediction of the time-space correlation using the model with finite Lagrangian correlation is calculated (ITLT2). Fig. 5 shows the predictions for this flow configuration. While TH again simply shifts the autocorrelation peak, ITLT again shows it’s complex behavior as before. However, this time ITLT predicts too an large correlation peak because of the assumption of infinitely long Lagrangian correlation, which is not the case in this experiment. The new ITLT2 considers the decay of the Lagrangian correlation and it’s prediction is much closer to the experimental CCF values. However, a detailed view onto the shape of the predicted and the measured correlation peaks discovers deviations, which indicate that the introduced advection model is not capable to reproduce all characteristics of the real process.

6. Conclusions

The first results of the new model to transform Eulerian temporal correlation functions into longitudinal spatial correlation and time-space correlations for finite Lagrangian correlation are promising and show a perspective to predict statistics of turbulent flows in an intermediate range of scales, where complex interactions between small turbulent scales and large scale structures happen. However, deviations between the predicted and the measured correlations indicate limitations of either this advection model or the transformation or both. A further improvement of the transformation requires deeper understanding of the transport process of coherent structures in turbulent flows and a proof of the mapping from one correlation to another. In the present paper the Eulerian correlation function has been transformed directly into a time-space correlation function based on the consideration of the Lagrangian correlation function. However, in reality, both, the Eulerial correlation function and the time-space correlations are projections of the Lagrangian correlation function, such that the transformation possibly must be refined in the future.

References


![Fig. 1 Experimental setup for two-point cross-correlation measurements on the centerline of a round free air jet by laser Doppler velocimetry]

**Table 1** Flow specifications

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer diameter $d_o$</td>
<td>140 mm</td>
</tr>
<tr>
<td>Inner diameter $d_i$</td>
<td>8 mm</td>
</tr>
<tr>
<td>Outer velocity (at nozzle exit)</td>
<td>0.5 m/s</td>
</tr>
<tr>
<td>Inner velocity (at nozzle exit)</td>
<td>35.9 m/s</td>
</tr>
<tr>
<td>Outer volume flux</td>
<td>27.6 m³/h</td>
</tr>
<tr>
<td>Inner volume flux</td>
<td>6.5 m³/h</td>
</tr>
<tr>
<td>Kinematic viscosity $\nu$</td>
<td>$14 \times 10^{-6}$ m²/s (air)</td>
</tr>
<tr>
<td>Reynolds number $Re = \frac{U_0d_i}{\nu}$</td>
<td>$2 \times 10^4$</td>
</tr>
</tbody>
</table>

**Table 2** Specifications of the laser Doppler system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Laser</td>
<td>Ar, max. 5 W, used with 5 mW, multi-mode</td>
</tr>
<tr>
<td>Wavelength</td>
<td>514.5 and 488 nm</td>
</tr>
<tr>
<td>Optical configuration</td>
<td>Fiber-coupled probe, backscatter</td>
</tr>
<tr>
<td>Frequency shift</td>
<td>Bragg cell, 40 MHz</td>
</tr>
<tr>
<td>Focal length</td>
<td>310 mm</td>
</tr>
<tr>
<td>Measurement volume</td>
<td>400 µm × 50 µm</td>
</tr>
<tr>
<td>Processor</td>
<td>IFA750</td>
</tr>
</tbody>
</table>
Fig. 2 Comparison of the two predicted time-space correlation functions for a separation of $\xi=32$ mm (4$d$):
ACF: autocorrelation function obtained from the one-point measurements. TH: Taylor’s frozen flow hypothesis. ITLT: integral time-length transformation. CCF: cross-correlation functions from two-point measurements.

Fig. 3 Fourier modes of the spatial transfer functions for a separation of $\xi=32$ mm (4$d$): ACF+TH: Taylor’s frozen flow hypothesis from autocorrelation function. ACF+ITLT: integral time-length transformation from autocorrelation function. CCF: cross-correlation functions from two-point measurements.

Fig. 4 Integral time scales obtained from the cross-correlation function of experimental two-point measurements as a function of the separation of the measurement volumes.
Table 3 Flow specifications (experiment 2)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outer diameter $d_o$</td>
<td>55 mm</td>
</tr>
<tr>
<td>Inner diameter $d_i$</td>
<td>6 mm</td>
</tr>
<tr>
<td>Inner velocity (at nozzle exit) $U_0$</td>
<td>44.5 m/s</td>
</tr>
<tr>
<td>Kinematic viscosity $\nu$</td>
<td>$14 \times 10^{-6}$ m/s (air)</td>
</tr>
<tr>
<td>Reynolds number $Re = \frac{U_0 d_i}{\nu}$</td>
<td>$1.9 \times 10^4$</td>
</tr>
</tbody>
</table>

Fig. 5 Comparison of the two predicted time-space correlation functions for the second experiment with decaying Lagrangian correlation at a separation of $\xi=40$ mm ($6.7 d_i$): ACF: autocorrelation function obtained from the one-point measurements. TH: Taylor’s frozen flow hypothesis. ITLT: integral time-length transformation. ITLT2: integral time-length transformation for decaying Lagrangian correlation. CCF: cross-correlation functions from two-point measurements.