Pressure-Field Extraction from Lagrangian Particle Tracking using a Voronoi Tessellation-Based Algorithm

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Abstract A new technique is described for pressure extraction from Lagrangian particle-tracking data. The technique uses a Poisson solver to extract the pressure field on a network of data nodes constructed using the Voronoi tessellation and the Delaunay triangulation. This technique is validated using an analytical test case based on potential flow around a circular cylinder, and is demonstrated on particle-tracking data obtained for the flow around an impulsively-started NACA 0012 wing with an aspect ratio of $\text{AR}=4$, an angle of attack of $\alpha=45^\circ$, and a Reynolds number of $\text{Re}=7500$. In the analytical validation, the pressure field extracted from the synthetic data closely matches the analytically calculated pressure; however, a decrease in accuracy is observed near the surface of the cylinder. This is assumed to be caused by the combination of a mismatch of the number of particles in the field and the number of particles placed on the surface, and by the difficulty in enforcing the Neumann condition properly with unstructured data. The dependence of average field error and computational time on the number of field nodes is examined using the analytical test case. The average field error is assumed to be a superposition of truncation error and rounding error, causing the error to first exponentially decrease, then logarithmically increase, with additional nodes. The relationship between computational time and the number of nodes is found to follow a power law, where the power was 1.5. In the analysis of the experimental data, the extracted pressure field is evaluated based on how well the pressure minima highlight the leading-edge and tip vortices. The sensitivity of the pressure field to additional nodes is calculated for several different particle densities. It is seen that having between 60000 and 90000 nodes appears to be a critical particle-density range for the current problem. In this range both the leading-edge and tip vortices are clearly identified by local pressure minima, and the pressure field in the nominally two-dimensional region one chord in-board of the wing assumes its qualitatively-expected form. Additionally, from 60000 to 90000 nodes there is a half order-of-magnitude decrease in the sensitivity of the pressure field to additional nodes. It is assumed that the spatial node density required for accurate pressure extraction will scale linearly with the Reynolds number for nominally two-dimensional flow structures and with the square of Reynolds number for fully three-dimensional flow structures. Therefore, the computational time for pressure extraction will scale with the power of 1.5 and 3 of Reynolds number for nominally two-dimensional and fully three-dimensional cases, respectively.

1 Introduction

1.1 Background
As the feasibility and prevalence of Lagrangian flow measurement techniques increase, it is imperative that the possible applications of Lagrangian data are fully explored. Previous studies have investigated pressure field extraction using optical flow measurements; however, this work has focused on extracting pressure from Eulerian flow data [1, 2]. More recently, Violato et al. investigated a pressure extraction technique performed on a structured grid where the Lagrangian material derivative field was found by reconstructing particle trajectories using tomographic-PIV data [3]. Novara et al. investigated pressure extraction by combining tomographic-PIV measurements with three-dimensional particle tracking velocimetry in order to substantially increase the particle density of the Lagrangian measurements, but the pressure extraction was ultimately performed by interpolating the Lagrangian material derivative field to a structured grid [4]. A particularly compelling use of Lagrangian data that has yet to be investigated is the extraction of pressure fields entirely in the Lagrangian frame. Unlike
Eulerian frameworks, in which data nodes remain temporally fixed in space, the nodes in Lagrangian frameworks displace with the flow and thus are positioned arbitrarily in a given frame. The primary advantages of operating in a Lagrangian framework are that the errors inherent in interpolating Lagrangian data onto a grid are avoided, the material derivative is simple to calculate, and it is easier to obtain reliable data close to physical boundaries compared to purely Eulerian measurement techniques [5]. One disadvantage of a Lagrangian framework is the lack of consistency in spatial density of data, which can lead to difficulty evaluating spatial derivatives accurately. While the spatial density of Lagrangian particle tracking data has historically been inferior to tomographic-PIV, recent advancements in processing techniques will allow Lagrangian particle tracking data to be collected at spatial densities equal to if not greater than that of tomographic-PIV [6, 7].

1.2 Poisson Equation for Pressure

In previous studies, two distinct methodologies have been employed in extracting the pressure field from Eulerian flow data. The pressure field can be calculated by converting the Navier-Stokes equation to a Poisson equation and extracting the pressure field by integration of the Poisson equation [8]. Alternately, the pressure gradients can be calculated using the vector Navier-Stokes equations and directly integrated to find the pressure field [9]. Each of the pressure extraction techniques requires the evaluation of a third-order derivative. In the gradient integration technique the third-order derivative is the Laplacian of velocity, which is a second-order spatial and first-order temporal derivative, while in the Poisson equation it is the divergence of the material derivative, which is a first-order spatial and second-order temporal derivative. In a Lagrangian frame, temporal derivatives are much more robust than spatial derivatives, since temporal derivatives are simply calculated along a particle track while spatial derivatives must be calculated on an unstructured network of nodes. The emphasis of temporal derivatives over spatial derivatives in the Poisson solver, combined with the robustness of Lagrangian temporal derivatives and the difficulty in calculating unstructured spatial derivatives, indicates that a Poisson solver is more suited to Lagrangian pressure extraction than a gradient integration method. In addition, the Poisson solver technique has been shown to be less prone to error propagation when compared to the technique of pressure gradient integration [3]. Therefore a Poisson solver is applied to Lagrangian particle tracking data in the present study.

The general incompressible form of the Navier-Stokes equation in vector notation is:

\[ \nabla p = -\rho \frac{D\vec{v}}{Dt} + \mu \nabla^2 \vec{v} + \vec{F}_b, \]

where \( \nabla p \) is the gradient of pressure, \( \frac{D\vec{v}}{Dt} \) is the material derivative of velocity, \( \nabla^2 \vec{v} \) is the Laplacian of velocity, \( \vec{F}_b \) are body forces, \( \rho \) is the density, and \( \mu \) is the dynamic viscosity. Applying the divergence operator to both sides of equation (1), we obtain a single differential expression for pressure:

\[ \nabla^2 p = -\rho \left( \nabla \cdot \frac{D\vec{v}}{Dt} \right), \]

where the divergence of the viscous term is eliminated by a vector identity combined with the assumptions of incompressibility and continuity, and the divergence of the body force is assumed to be negligible.

In a Lagrangian framework, the material derivative can be evaluated using only a temporal derivative along the particle track:

\[ \frac{D\vec{v}}{Dt} = \frac{d}{dt} \vec{v}(\vec{x}_p(t)) = \frac{d^2}{dt^2} \vec{x}_p(t), \]

where \( \vec{v}(\vec{x}_p(t)) \) is the velocity of particle \( p \) at position \( \vec{x}_p(t) \) and time \( t \). It is evident that the evaluation of the material derivative in a Lagrangian framework is a simple matter of evaluating the derivative of velocity along a pathline. With this, the final form of the Poisson equation for pressure can be written as:

\[ \nabla^2 p = -\rho \left( \nabla \cdot \frac{d^2}{dt^2} \vec{x}_p(t) \right). \]
Equation (4) is the Poisson equation for pressure, with the source function given as the negative of density multiplied by the divergence of the acceleration of the fluid.

Figure 1: Example of a Voronoi tessellation in a two-dimensional space. The black dots represent data node locations. Black lines indicate connections between neighbours, red lines indicate Voronoi cell boundaries. Point $i$ and neighbour $j$ have a corresponding distance and Voronoi cell face size, labeled $h_{ij}$ and $s_{ij}$ respectively.

1.3 Meshing the Domain
To numerically solve the Poisson equation and extract pressure, vector-calculus operators such as the divergence, gradient, and Laplacian are necessary. In an Eulerian frame the evaluation of these operators is trivial, as standard finite difference equations can be used. In a Lagrangian frame, a network must be constructed on the field of data to perform vector-calculus operations. In the present study the network is constructed using the Voronoi tessellation and its dual, the Delaunay triangulation.

In three-dimensional space, the Delaunay triangulation of a field of data nodes meshes the domain with tetrahedra, where each tetrahedra is a quadruplet of data nodes and the circumsphere of the tetrahedra does not contain any other data nodes. Any two points which are in the same tetrahedra are then considered to be neighbours in the network. The Voronoi tessellation then creates a set of cells that correspond bijectively to the set of data nodes. Each Voronoi cell is a polyhedron whose faces correspond bijectively to the neighbours of the data node associated with the Voronoi cell. An example of a Voronoi cell in two-dimensional space is shown in figure 1.

1.4 Numeric Solution of the Poisson Equation in an Unstructured Domain
The three vector calculus operators needed to numerically solve the Poisson equation are the gradient operator, the divergence operator, and the Laplace operator. An expression for evaluating the gradient operator in an unstructured discrete environment can be derived by applying Green’s theorem to a general scalar field and discretizing the result with respect to the parameters of the network. An expression for the divergence can be derived by applying the same process to a general vector field. The resulting expressions are shown below:

$$\vec{\nabla}f_i = \frac{\sum_j [(f_i + f_j) s_{ij} \hat{n}_{ij}]}{\frac{1}{D} \sum_j (s_{ij} h_{ij})},$$

(5)

$$\vec{\nabla} \cdot \vec{F}_i = \frac{\sum_j [s_{ij} \hat{n}_{ij} \cdot (\vec{F}_i + \vec{F}_j)]}{\frac{1}{D} \sum_j (s_{ij} h_{ij})},$$

(6)
where \( f \) is some scalar field, \( \vec{F} \) is some vector field, \( D \) is the dimension of the space, and \( \hat{n}_{ij} \) is the normalized vector pointing from \( i \) to \( j \). For the Laplace operator, the following equation is used [10]:

\[
\nabla^2 f_i = \sum_j \left( \frac{s_{ij}}{h_{ij}} f_j \right) - f_i \sum_j \left( \frac{s_{ij}}{h_{ij}} \right) \frac{1}{2D} \sum_j (s_{ij}h_{ij}).
\]  

(7)

Equation (5) can be rearranged to isolate for the value of the field at a node and is used to enforce a Neumann boundary condition, while equation (6) is used to compute the value of the source field. Rearranging equation (7) to isolate for the value of the scalar field at point \( i \) provides an equation that is used to solve the Poisson equation iteratively on interior nodes. The iterative integration technique used is known as the Successive Over-Relaxation (SOR) algorithm, which is a method which can be used to stabilize a diverging iterative process or speed up the convergence of a slowly converging iterative process. If the next iteration of \( \phi \) is given by:

\[
\phi^{[k+1]} = f \left( \phi^{[k]} \right),
\]  

(8)

where \( \phi \) is a variable or field and \( f \) is some function, then the SOR algorithm for iteratively solving for \( \phi \) is:

\[
\phi^{[k+1]} = (1 - \omega) \phi^{[k]} + \omega f \left( \phi^{[k]} \right),
\]  

(9)

where \( \omega \) is the relaxation parameter. Values of \( \omega > 1 \) cause the convergence rate to increase but can induce instability, while values of \( \omega < 1 \) will slow down the rate of convergence while making the process more stable.

2 Analytical Test Case

2.1 Potential Flow around a Circular Cylinder

Potential flow around a circular cylinder was chosen for initial validation of the technique for two primary reasons. First, potential flow around a cylinder is one of the simplest available analytical solutions to a flow field, which makes it ideal for an initial assessment of a novel analysis technique. Second, there is an impermeable...
surface in the flow field as well as multiple open boundaries with varying conditions, which is a similar set of boundaries that would be present in typical optical flow-measurement datasets. With a priori analytical knowledge of the pressure field, the pressure-extraction error can be precisely quantified. Extracting pressure fields from synthetic data allows assessment of the relationship between pressure extraction error and spatial particle density in the absence of uncertainty in the velocity data. The potential function governing flow around a cylinder is:

$$\phi (r, \theta) = U_\infty \left( r + \frac{R^2}{r} \right) \cos (\theta),$$  \hspace{1cm} (10)

where $r$ is the radial position, $\theta$ is the angular position, $U_\infty$ is the free-stream velocity, and $R$ is the radius of the cylinder. The radial and tangential velocities are then:

$$U_r (r, \theta) = \partial_r \phi (r, \theta) = U_\infty \left( 1 - \frac{R^2}{r^2} \right) \cos (\theta),$$  \hspace{1cm} (11)

$$U_\theta (r, \theta) = r^{-1} \partial_\theta \phi (r, \theta) = -U_\infty \left( 1 + \frac{R^2}{r^2} \right) \sin (\theta).$$  \hspace{1cm} (12)

The pressure field around the cylinder is calculated using Bernoulli’s equation and is given by:

$$p (r, \theta) = \frac{1}{2} \rho U_\infty^2 \left( 2 \frac{R^2}{r^2} \cos (2\theta) - \frac{R^4}{r^4} \right) + p_\infty.$$  \hspace{1cm} (13)

Finally, when discussing error and generalizing the results of the assessment of this pressure extraction technique, it is useful to discuss pressure in a dimensionless form, for which the pressure coefficient can be used. The expression for the pressure coefficient around the cylinder is:

$$C_p (r, \theta) = \frac{p (r, \theta) - p_\infty}{\frac{1}{2} \rho U_\infty^2} = 2 \frac{R^2}{r^2} \cos (2\theta) - \frac{R^4}{r^4}.$$  \hspace{1cm} (14)

2.2 Analytical Results

To assess the accuracy of the pressure extraction technique, synthetic Lagrangian datasets were created and processed. To reduce the computation time, synthetic particle locations were restricted to a cylindrical region around the cylinder which was $4R$ in diameter. Using equations (11,12), the synthetic particles were assigned velocity and material derivative values such that the field of data nodes was equivalent to typical Lagrangian particle tracking output. A Dirichlet boundary condition was used at the outer boundary of the synthetic measurement region by randomly distributing nodes along the outer boundary surface of the measurement volume. To prevent network links from crossing the cylinder, extra nodes were distributed on the cylinder surface that were networked only to field nodes. In addition to preventing the network from impinging on the cylinder, these nodes allowed extraction of the surface pressure on the cylinder. Once the synthetic Lagrangian data and boundary value nodes had been assembled, the nodes were networked and the pressure was extracted using the technique outlined in section 1. In the iterative integration stage, a relaxation parameter value of $\omega = 1.02$ was found to increase the rate of convergence of the solution without causing instability. The solution was considered to have converged when the average change in the pressure field across the domain was less than $10^{-6} C_p$, and all datasets were found to converge satisfactorily within 1000 iterations.

Figure 3(a) shows the relationship between the average pressure-coefficient error in the field and on the cylinder compared to the analytical pressure field, and the number of data nodes in the measurement region. The pressure extraction error initially decays exponentially with increasing quantities of nodes in the measurement volume, but after reaching a minimum it begins to increase logarithmically. The proposed mechanism for this behaviour is a superposition of two separate sources of error: under-sampling error and rounding errors. Under-sampling error would be dominant for smaller quantities of nodes and decay as the number of nodes increased.
Rounding errors could be caused by large variations of Voronoi cell-face sizes and neighbour distances. The distribution of data nodes is entirely random and as such the number of nodes that will suffer from rounding errors will increase with the number of data nodes. This proposed mechanism continues to be investigated and a definitive explanation for the relationship between pressure extraction error and quantity of data nodes has yet to be established. The pressure extraction errors in the field and on the cylinder show the same general behaviour, but the average error is larger on the surface. The minimum error and inflection point occur at larger quantities of data nodes for the pressure on the surface than for the pressure in the field. An additional predicted mechanism of this behavior for the surface nodes is the possibility of network links forming through the cylinder at very high numbers of data nodes.

It can be seen in figure 3(b) that the relationship between the integration time and the quantity of data nodes is linear. The relationship between the computational time of the Voronoi network construction and the quantity of data nodes is a power relationship, where the power is approximately 1.5. In cases where the flow field is highly two-dimensional, it would be reasonable to assume that the number of nodes required to accurately extract the pressure field will scale linearly with the Reynolds number. When applied to fully three-dimensional cases, it is assumed that the number of nodes will scale with the square of Reynolds number. Therefore, it can be assumed that the computational cost of using this technique will scale at worst with the Reynolds number to the power of 1.5 and 3, for two-dimensional and three-dimensional cases, respectively.

Figure 4 shows, from left-to-right, sample plots of the extracted pressure field and the error in the extracted pressure field for 3000, 30000, and 60000 nodes. While the pressure extraction was performed in three dimensions, the pressure and velocity fields are independent of the spanwise dimension and so for convenience all data nodes are displayed in the streamwise-transverse plane only. It can be seen in all three cases that the error in pressure extraction is greatest close to the surface of the cylinder. This is expected as pressure extraction is less accurate on the surface of the cylinder, and these errors will propagate outward due to the way the integration is performed. In all three cases the overall pressure-field estimation is accurate to within 0.1 $C_p$. In figure 4(e) it can be seen that the highest errors occur at the surface in the low-pressure region, where the sink term in the Poisson equation is greatest. The scattered locations of largest pressure extraction errors support...
the assumption that the increase in error is due to random rounding errors. The increased error in the vicinity of the surface indicates that further investigation is required into the optimal number of surface data nodes for a given number of field nodes.

![Figure 4: Coloured scatter plots of (top) extracted pressure fields and (bottom) the error in the extracted pressure field as compared to the analytically-known pressure field, both shown in terms of the pressure coefficient. From left to right, the number of nodes in the measurement volume increases: (left) 3000 nodes, (middle) 30000 nodes, (right) 60000 nodes.](image)

3 Experimental Data

3.1 Experimental Methods

Experiments were performed in a free-surface towing tank at the University of Calgary. Experiments were performed by using a traverse system to impulsively accelerate a NACA 0012 wing, with aspect ratio of $AR = 4$ and an angle of attack of $\alpha = 45^\circ$, from rest to a final velocity of $U_f = 0.1 \text{ m/s}$ as it traveled through the measurement region. Figure 5 shows two views of the position of the NACA 0012 at a particular position as it passes through the measurement volume, and table 1 shows a list of the relevant parameters of the experiment.

<table>
<thead>
<tr>
<th>profile</th>
<th>NACA 0012</th>
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<tr>
<td>medium</td>
<td>water</td>
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<tr>
<td>chord (m)</td>
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<td>acceleration period ($t^*$)</td>
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<tr>
<td>final velocity (m/s)</td>
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<td>temperature (°C)</td>
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<td>Reynolds number</td>
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Table 1: Parameters of the experimental case used for pressure extraction.
The towing tank was seeded with 100-µm hollow glass spheres, with a Stokes number of \( St = 2.4(10^{-3}) \), which were assumed to follow the flow. The towing tank was heated to 38°C to increase the Reynolds number while keeping the chord and final velocity within the limits imposed by the high-speed camera frame rate. With the increased water temperature, the Reynolds number of the experimental case was \( Re = 7500 \). The measurement region was limited to a circular column oriented in the span-wise direction with a length of \( l = 0.16 \text{ m} \) and a diameter of \( d = 0.08 \text{ m} \). The coordinate system was oriented such that the \( x \)-direction pointed in the stream-wise direction, the \( z \)-direction pointed in the span-wise direction, and the \( y \)-direction pointed in the transverse direction. Images recorded by the high-speed cameras were processed using a three-dimensional Lagrangian particle tracking technique to obtain Lagrangian flow data [11]. Time was non-dimensionalized using equation (15) such that the occurrence of flow developments could be properly compared between cases:

\[
t^* = \frac{t U_f}{c}.
\]  

Figure 6 shows the Lagrangian particle-tracking set-up. Four PCO.edge sCMOS high-speed cameras were used, each recording with a resolution of 2056×1280 pixels and at a frame rate of 120Hz. The experiment

Figure 5: NACA 0012 position in measurement volume in two separate planes. The transverse location of the wing was chosen for optimal optical access to the suction side of the flow field.

Figure 6: The PTV experimental set-up is shown with (left) the room lights on and HID light source off, and (right) the room lights off and HID light source on. (a) PCO.edge sCMOS camera array, (b) continuous HID light source, (c) NACA 0012 mounted on traverse with rotation stage, (d) measurement volume with NACA 0012 at starting position.
was repeated twenty times and the Lagrangian particle-track fields were phase-averaged together by combining them into a single Lagrangian dataset. In order to investigate the benefit of higher-density datasets than are possible with the currently-available PTV algorithms, time-averaging was also performed wherein the Lagrangian data from several time steps was compressed into a single dataset.

3.2 Experimental Results

The pressure field was extracted from the experimental data using the method outlined in section 1. A small Dirichlet boundary was imposed on the outboard edge of the measurement region. Data nodes that were found to be unbounded were treated as the exterior boundary nodes on the streamwise and transverse boundaries, as well as at the boundary created by the wing. A boundary was placed at the approximate location of the wing to prevent network links from forming through the solid boundary. The boundary points located near the surface of the wing were treated as interior nodes that were only connected to points on a single side.

Figure 7 shows the Lagrangian particle-track field around the NACA 0012 for \(0 < t^* < 0.5\). In order to highlight the leading-edge vortex and the tip vortex, displayed tracks were filtered based on dimensionless vorticity, defined as:

\[
\omega^* = \frac{\omega_c}{U_\infty}.
\]  

Figure 7(a) and Figure 7(b) show the leading-edge vortex and the tip vortex, respectively. It can be noted that some tracks appear to coincide with the surface of the wing in Figure 7(b), this is because the tracks are shown in a lab-fixed frame of reference, while the wing is shown in the position it occupies at \(t^* = 0.25\).

Figure 8 shows an isometric view of four pressure fields, all at the same point in time, \(t^* = 0.25\), but with increasing degrees of time-averaging used, varying from a single snapshot to the time-average of seven snapshots, and ranging from under 10000 nodes to just under 70000 nodes. The pressure fields are plotted as coloured three-
Figure 8: Isometric view of the pressure field for four different numbers of data nodes. (top left) $t^* = 0.250$, (top right) $t^* = 0.250 \pm 0.017$, (bottom left) $t^* = 0.250 \pm 0.033$, (bottom right) $t^* = 0.250 \pm 0.050$. $N$ is the number of data nodes in each time-averaged dataset. In all cases shown, the regions of low pressure correspond to the location of the leading-edge and tip vortices. As more nodes are added, the two vortices become increasingly resolved. For $N = 69114$ the low pressure core of the full leading-edge vortex tube is visible one chord in-board from the tip.
dimensional scatter plots where the colour shows $C_p$, which is the non-dimensional pressure value previously defined as the pressure coefficient in equation (14):

$$C_p = \frac{p - p_\infty}{\frac{1}{2}U_\infty^2 \rho}.$$  

The pressure field is filtered such that only nodes with a pressure coefficient of $C_p < -1$ were plotted so as to highlight regions that contained local pressure minima. The approximate location of the NACA 0012 wing is also shown in each plot, and is travelling in the negative $x$-direction at 0.1m/s.

The experimental dataset did not have a known pressure field with which the extracted pressure fields could be compared to evaluate the accuracy of the algorithm. Instead, qualitative criteria were used to evaluate the quality of the extracted pressure field, which were: the ability to identify the leading-edge vortex and tip vortex by viewing pressure minima, the qualitative features of the nominally two-dimensional region that begins one chord inboard of the tip and extends towards the free surface [12], and the sensitivity of the pressure field to the number of measured data nodes.

It can be seen in figure 8 that the minimum pressure occurs downstream and out-board of the wing along the leading edge and tip, in the same region where the leading-edge vortex and tip vortex are observed in figure 7. Using only the pressure field extracted without time averaging, the leading-edge vortex and tip vortex are difficult to identify, as there is a general region of low pressure but very few nodes in the vortex core. As the degree of time averaging and therefore the number of nodes increases, both the leading-edge vortex and tip vortex become much more clearly-defined. In the lower-right plot in figure 8, with approximately 70000 nodes, it is possible to observe the full leading-edge vortex tube simply by observing the pressure minima, due to the large number of nodes present in the vortex core.

Kreigseis et al. showed that the structure of the flow beyond one chord in-board from the tip of a finite aspect ratio wing is nominally two-dimensional for impulsive motions [12]. Therefore the region of flow beyond one chord in-board from the tip will be assumed to be nominally two-dimensional. Figure 9 shows the nominally two-dimensional region of the flow ($z/c > 1$) compressed into two-dimensional scatter plots, coloured by the Lagrangian pressure coefficient around the leading edge of the wing. Again, all four scatter plots are of the time-averaged pressure field centered on $t^* = 0.25$ and varying from a single snapshot in time to a time-average of seven time steps, ranging from under 10000 nodes to just under 70000 nodes. The pressure field is not filtered by any criteria; however, the size of the domain has been decreased to focus on the leading-edge vortex and surrounding fluid.

Similar to figure 8, it is difficult to identify the location of the leading-edge vortex using the dataset without any time-averaging, since the nodes are scattered and there is no clear pressure minimum. In addition, it can be seen that the pressure coefficient on the suction side of the wing does not appear to be lower than the free-stream fluid out-board from the leading edge. This qualitatively disagrees with the expected pressure-field distribution around a moving wing, wherein a pressure deficit would be expected on the suction side of the wing. As the degree of time-averaging and number of nodes increases, it can be observed that the pressure profile of the leading-edge vortex becomes more clearly-defined. Additionally, in the pressure field with approximately 70000 nodes, a pressure deficit can be clearly observed on the suction side of the wing compared to the out-board free-stream fluid.

Figure 10 shows the results of a sensitivity analysis of the experimental data. The sensitivity of the pressure field was defined as:

$$S(N) = \frac{\Delta C_p}{\Delta N},$$  

where $S(N)$ is the sensitivity of the pressure field extracted from $N$ nodes, $\Delta C_p$ is the field-averaged difference
in pressure coefficient, and $\Delta N$ is the difference in number of nodes. The relationship between the sensitivity and the number of nodes is unstable for $N<60000$. For $60000<N<90000$ there is a decrease in sensitivity with number of nodes, which appears to stabilize for $N>90000$. The general trend of decreasing sensitivity with increased numbers of nodes, and the absence of spikes in sensitivity after $N=30000$, indicates that the pressure field is converging towards the true value as nodes are added to the field. Additionally, the half order-of-magnitude drop of sensitivity from 60000 to 90000 nodes indicates that there may be some critical number of nodes that optimizes the ratio of pressure-field accuracy to computational time.

![Figure 9: Nominally two-dimensional region of pressure field, one chord inboard from tip. (top left) $t^*=0.250$, (top right) $t^*=0.250\pm0.017$, (bottom left) $t^*=0.250\pm0.033$, (bottom right) $t^*=0.250\pm0.050$. $N$ is the number of data nodes in each time-averaged dataset. For $N<10000$, the leading-edge vortex cannot be discerned, and there is no identifiable pressure deficit on the suction side of the wing compared to the free-stream. As more nodes are added to the field, the pressure distribution of the leading-edge vortex becomes more clearly defined, and for $N=69114$ the suction side of the wing displays a pressure deficit compared to the free-stream.](image-url)
Figure 10: Log plot of the sensitivity of the pressure field to changes in numbers of nodes versus the number of nodes in the field. Sensitivity of pressure field to number of nodes appears to be unstable prior to 60000 nodes. Between 60000 and 90000 nodes a half order-of-magnitude drop in the sensitivity of the pressure field to additional nodes is observed, which continues to decrease slightly up to 110000. This suggests that $60000 < N < 90000$ is a critical region, and that the pressure field is converging towards a final result as more nodes are added.

4 Conclusions

In the present study, a technique for extracting the instantaneous or short-interval, time-averaged hydrodynamic pressure field from Lagrangian particle-tracking data has been demonstrated. First, the technique was tested on a simple analytical case based on potential flow around a cylinder. The analytical test case served to prove the theoretical validity of the technique, to characterize the relationship between computation time and number of nodes, and provided a first impression of the relationship between extraction error and number of nodes. It was found that the relationship between computation time and number of nodes is a power relationship where the power is approximately 1.5. The relationship between extraction error was assumed to be a superposition of under-sampling, finite difference truncation error and rounding errors.

The technique was then demonstrated using experimental data. This data was obtained by measuring the Lagrangian flow field around the leading edge and tip of a NACA 0012 profile wing, and pressure extractions were performed on both individual snapshots in time and on short-interval time-averaged particle fields. The analysis of the experimental data served to prove that with sufficient particle density it is feasible to use the technique to extract time-averaged pressure fields. However, it reinforced the conclusion from the analytical test case that pressure extraction on or near a solid boundary is more difficult, due to the difficulty in properly implementing boundary conditions at walls and the decrease in density of data nodes near surfaces.

From the analysis of the experimental results, it appears that 60000 to 90000 nodes is a critical range for pressure extraction. In figure 8, it was observed that for approximately 70000 nodes the full leading-edge vortex tube was fully highlighted by local pressure minima. In figure 9, it was observed that for 70000 nodes the leading-edge vortex is clearly defined and the suction side of the wing is at a negative pressure relative to the out-board fluid in the free-stream. Additionally, the analysis of the experimental data shows a half order-of-magnitude decrease in the sensitivity of the pressure field to added nodes in the range $60000 < N < 90000$. Recently, Schanz et al. demonstrated a new technique for three-dimensional PTV with which it is possible to perform accurate PTV measurements with up to 70000 particles in a single frame without phase-averaging or time-averaging [6]. This new technique would allow for instantaneous pressure-field extraction, with no phase-averaging, for cases with similar Reynolds numbers and measurement volume size as the one discussed in the present study.
The number of measured particles per frame required for accurate pressure-field extraction will scale linearly with the ratio of the measurement domain volume to the characteristic volume scale of the problem. It will also scale linearly and quadratically with the Reynolds number for nominally two-dimensional cases and fully three-dimensional cases, respectively. Since the computational time needed to create the Voronoi network necessary for Lagrangian pressure extraction scales with the number of nodes to the power of 1.5, the computational time needed for accurate pressure extraction will scale with the Reynolds number to the power of 1.5 and 3 for nominally two-dimensional flows and fully three-dimensional flows, respectively.

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