The Identification of Lagrangian Coherent Structures in a Nominally Two-Dimensional Shear Flow via Lagrangian Particle Tracking

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Abstract: The quantification of Lagrangian Coherent Structures (LCS) has been investigated using a novel algorithm based on the tessellation of unstructured data points. The applicability of such an algorithm in resolving an LCS was first tested using a synthetically-generated double-gyre flow with a known LCS solution. Subsequently, the algorithm was tested on experimental data acquired using two-dimensional particle tracking velocimetry (2D-PTV) of a nominally two-dimensional free shear flow. To quantify factors linked to accurate LCS identification, dimensionless parameters are developed to characterize their relationship to LCS resolution. Specifically, the study considers turnover number ($TN$), which is defined here as the fraction of the particle track length to the vortical time scale, and cell ratio ($CR$), which can be thought of as the relative density of particles with long track lengths per characteristic vortex length scale. When examining the double-gyre finite-time Lyapunov exponent (FTLE) field, as expected, the resolution of the well-defined ridge improved with increasing $TN$. Upon use of tracks lengths >30% of the overall vortex time scale, only modest improvement in ridge resolution was observed. The LCS of the experimental data set was well-resolved using track lengths on the order of 20-30% of the vortex time scale. However, the use of greater $TN$ values presented LCSs with an increasingly disordered coherence suggesting an inadequate particle count to accurately define the structure. Additionally, an exponential relationship was observed between $CR$ and $TN$. This suggests that a compromise must be made between track length and the density of tracked particles to adequately resolve an LCS where an insufficient number of particles have long track lengths.

1 Introduction

The ability to identify independent regions of coherent motion, or coherent structures, within temporally and spatially resolved flow data is essential in the study of unsteady flows. Technological limitations of early optical measurements have resulted in the acquisition of flow fields using an Eulerian framework. As a consequence, traditional methods of coherent-structure identification relies on the input of Eulerian data. The original Eulerian-based technique was the Q-criterion developed by Hunt et al. [1]. More recent techniques include the $\Delta$-criterion developed by Chong et al. [2], and the swirling-strength and $\Gamma$ criteria, both developed specifically for vortices by Zhou et al. [3] and Graftieaux et al. [4], respectively. The commonality between all Eulerian identification methods is their reliance on the decomposition of the velocity-gradient tensor $\nabla v$ into the summation of the rate-of-strain and vorticity tensors, $\Omega$ and $S$, respectively. The eigenvalues of $S$ and $\Omega$, and consequently the coherent structures that the above techniques would identify, are invariant with Galilean-frame changes.

Although Eulerian-based coherent-structure identification is widely used, Eulerian coherent structures (ELSs) suffer from three deficiencies. Firstly, ELSs are reference-frame dependent whereupon a change would be observed for a transformation that involves acceleration; see Haller [5]. ELSs are also subject to user-defined thresholds leading to high subjectivity in structure boundary identification; see Green et al. [6]. Finally, the generation and breakdown of ELSs is purely speculative since one lacks the flow history of the fluid that comprises the structure. In contrast, coherent structures identified from Lagrangian data, known as Lagrangian coherent structures (LCSs), do not suffer from the aforementioned deficiencies. LCSs are identified as regions of extrema or inflection within a frame-invariant quantification of flow-map strain, which results in the identification of structures unaffected by frame transformations, eliminating the need of a user-defined threshold. Additionally, Lagrangian data provides the flow history of particles, which in turn yields further insight into
the generation and breakdown of coherent structures. Current state-of-the-art methods of LCS identification involve using the smallest eigenvalue of the Cauchy-Green strain tensor (Peacock and Haller [7]) or the identification of “black-hole” vortices, which are regions contained by manifolds of invariant tangential strain; see Haller [8].

Regardless of the selected LCS identification method, the data at one’s disposal must be comprised of tracks of equal length that start and end concurrently, providing temporally homogeneous data. Acquiring such data from experiments is currently unfeasible, thus making the identification of LCSs from experimentally acquired Lagrangian data challenging. The inherent inhomogeneity of experimental Lagrangian data sets results in a coarse strain field, which in turn can obscure the LCSs contained therein. In the particular case of two-dimensional particle tracking velocimetry (2D-PTV), the method is further limited by out-of-plane particle motion that causes pathline breakage; see Raben et al. [9] and Brunton and Rowley [10]. Three-dimensional measurement techniques can potentially reduce pathline-breakage effects, given that the out-of-plane particle motion can be accurately quantified. However, there are numerous environments where the acquisition of three-dimensional Lagrangian measurements are highly challenging or unfeasible. Thus, it would be advantageous to discern when the three-dimensionality of nominally two-dimensional flows are negligible such that two-dimensional Lagrangian measurements are capable of accurately capturing LCS structures.

Traditional methods of LCS identification from finite-time Lyapunov exponents (FTLE) are associated with high computational cost. Raben et al. [9] demonstrated that particle-tracking flow-map compilation (FMC), which utilizes recorded particle images to represent Lagrangian flow tracers, generates more accurate estimates of the FTLE flow field, specifically when seeding density is low. Opposed to past approaches that utilize velocity-field integration, FMC is computationally more accurate through elimination of numerical integration. However, to compute FTLEs, the FMC approach requires interpolation of irregularly sampled spatial data onto a rectilinear grid. With that in mind, the main thrust of the present study is the development of an FTLE computation method that uses unstructured Lagrangian data and excludes the use of interpolation to quantify an LCS using two-dimensional measurement techniques in a nominally two-dimensional flow. The objectives of the present study are two-fold. First, we wish to test the performance of our novel LCS identification algorithm that incorporates the use of unstructured Lagrangian data. Opposed to traditional approaches that rely on fixed, structured grid interpolation, we present a new algorithm that utilizes Lagrangian velocity data in its unstructured spatial form to quantify an LCS through FTLE computation. To verify this approach, our developed algorithm is first tested using a synthetically-generated flow field with a known LCS solution. Subsequently, our algorithm is applied to experimental data acquired from flow generated by a towed knife-edged plate through quiescent fluid in a free-surface water channel.

The accurate resolution of an LCS from experimental data is predicated on particle track length and the number of particles that present with such lengths. Ideally, an infinite number of particles would be tracked with a length equal to the full time domain of the measurement. However, in any experiment, the number of particles that are tracked across a specific threshold time is both finite and decays exponentially as the track length threshold is increased. This exponential decay, from here forward referred to as particle dropout, was presented by Schanz et al. [11] in reference to their "Shake-The-Box" evaluation approach to particle-based tomographic data using a three-dimensional measurement approach. It is intuitive then that this exponential particle-dropout decay would be more acute using two-dimensional measurement approaches such as that used here. Furthermore, the approach of Schanz et al. [11] permits the tracking of individual particles at pixel densities associated with tomographic PIV, increasing the number of available particles that present with long track lengths over traditional PTV approaches. Although an attractive measurement approach, imaging at such densities is not feasible in certain environments. We fully expect that our experimental data will present with few particle tracks that span the full time domain of our measurement. It is our second objective then to quantify the ability to characterize an LCS using Lagrangian experimental data upon the modification of particle track length and correspondingly, particle count. This will be achieved using a sensitivity analysis through the modification of track length to identify the minimum, non-dimensional track length required to quantify the LCS of our nominally two-dimensional experimental flow.

The following describes our developed method of LCS quantification using a spatially unstructured grid and the subsequent application of this algorithm to both synthetically-generated and experimental data sets. First,
a detailed overview of the method is described followed by a description of both the synthetic data set with a known LCS solution and the experimental methods used to acquire 2D-PTV amenable images. Detailed analysis and demonstration of this approach to double-gyre and experimental vortical flows is presented along with a sensitivity analysis of LCS identification as a function of dimensionless track length.

2 Methods

The current section explains the workings of our LCS algorithm using unstructured spatial data, followed by a description of our simulated and experimental test cases.

2.1 FTLE Computation

The FTLE measures the maximum linearized growth rate of distance between initially adjacent particles tracked over a finite time period. This requires the calculation of the Cauchy-Green ($C_{ij}$) deformation tensor, which is defined as:

$$C_{ij} = (\nabla \Phi_{t_0 + T}^T)^* \nabla \Phi_{t_0 + T}$$

where $\nabla \Phi_{t_0 + T}$ is the gradient of the flow map $\Phi_{t_0 + T}$ that maps fluid elements tracked at time $t_0$ to their final position at time $t_0 + T$ via Lagrangian particle tracking. Note that $(\nabla \Phi_{t_0 + T}^T)^*$ denotes the transpose of $\nabla \Phi_{t_0 + T}$. In two-dimensional flow, $\nabla \Phi_{t_0 + T}$ can be represented as:

$$\nabla \Phi_{t_0 + T} = \begin{bmatrix} \frac{dX_{t_0 + T}}{dx_{t_0}} & \frac{dY_{t_0 + T}}{dx_{t_0}} \\ \frac{dX_{t_0 + T}}{dy_{t_0}} & \frac{dY_{t_0 + T}}{dy_{t_0}} \end{bmatrix}$$

where the vectors $[X_{t_0}, Y_{t_0}]$ and $[X_{t_0 + T}, Y_{t_0 + T}]$ represent the original and final positions of the particles. Subsequently, identification of the highest eigenvalue ($\lambda_{max}$) from the deformation tensor corresponds to maximum stretching whereupon the FTLE field ($\sigma$) is given by:

$$\sigma_{t_0 + T} = \frac{1}{|T|} \ln \left( \sqrt{\lambda_{max} (C_{ij})} \right).$$

Computed FTLE maxima identified as distinct ridges are used to quantify the LCSs, which represent barriers to fluid flow that partition regions with unique dynamical behaviour (Espa et al. [12]).

2.2 LCS Determination from Unstructured Spatial Data

The successful identification of LCSs depends on the accuracy in calculating the spatial gradient of $\nabla \Phi_{t_0 + T}$; see Eq. 2. Thus, the ability to calculate spatial gradients is a necessary precursor to the successful identification of LCSs using unstructured data. The following describes the development of such an algorithm to calculate the necessary spatial derivatives.

A schematic representative of a PTV time step is presented in Fig. 1(a). At the current time step ($t_0$), $N$ particles located at positions $[X_{t_0}, Y_{t_0}]$ are tracked and an associated arbitrary scalar ($\Lambda$) is measured. From the data, the spatial gradient of $\Lambda$ ($\nabla (\Lambda)$) is calculated. Voronoi tesselation is performed towards this end, which results in a collection of polygon cells that each enclose a single particle located at the centroid of the cell; see Fig. 1(a). Fig. 1(a) focuses on a single particle ($P$) and its respective cell (II) outlined in red. II in this instance represents a bounded cell that is surrounded by particles in such a manner that all cell faces have finite lengths. Along the periphery of any flow field, there will unavoidably exist particles that are insufficiently surrounded by other particles, referred to as unbounded cells where one or more faces have infinite lengths. Bounded and unbounded cells are represented by filled and unfilled circles respectively in Fig. 1.
We focus specifically on the geometry of $\Pi$ and its relation to its neighbouring particles and cells. Fig. 1(b) shows both $P$, $\Pi$ and neighbouring particles of $P$. $\Pi$ presents with an area ($\Omega$) and consists of $J$ faces and $J$ neighbouring cells that each enclose an individual particle. The neighbouring particles of $P$, as well as the faces that cell $\Pi$ share with its corresponding neighbouring cells are enumerated by $I$. Thus, the lengths of the faces shared by cell $\Pi$ and cells $I$ and the distance between particle $P$ and particles $I$ are donated by $l_I$ and $d_I$, respectively.

If $P$ is surrounded by an infinite number of particles, $\Pi$ would form a smooth contour. Also, $\nabla \Lambda$ would be related to $\Lambda$ along contour $\Pi$ through the divergence theorem:

$$\int_\Omega \nabla(\Lambda) \cdot d\Omega = \oint_\Pi \Lambda \hat{n} d\Pi$$

(4)

where $\hat{n}$ is a unit vector pointing normal and outwards from $\Pi$. It is assumed here that cell $\Pi$ is sufficiently small such that $\nabla \Lambda$ is constant throughout $\Omega$. This assumption simplifies Eq. 4 to:

$$\nabla(\Lambda) = \frac{1}{\Omega} \oint_\Pi \Lambda \hat{n} d\Pi.$$  

(5)

Since the data is discrete, the path integral is replaced by a sum along all $J$ faces of $\Pi$, i.e.:

$$\oint_\Pi \Lambda \hat{n} d\Pi = \sum_{I=1}^{J} \Lambda_{F,I} \hat{n}_I l_I$$

(6)

where $n_I$ is a unit vector pointing normal to face $I$ and $\Lambda_{F,I}$ is $\Lambda$ evaluated along face $I$. A similar result is presented in Hyman et al., [13]. It is assumed that all $l_I$ are sufficiently small such that $\Lambda$ may be assumed constant along each face. $\Lambda$ at each face is therefore approximated as a linear interpolation between $\Lambda_P$ at $P$ and $\Lambda_I$ at particle $I$. Given the inherent geometry of the Voronoi cells, this linear approximation simplifies to $\Lambda_{F,I} = \frac{1}{2}(\Lambda_P + \Lambda_I)$. Furthermore, $\Omega$ can be divided into $J$ triangles where the vertices of each triangle are located at $P$ and at the end points of a single face $I$. This allows $\Omega$ to be represented as $\Omega = \sum_{I=1}^{J}(\frac{1}{4}l_I d_I)$. 

Figure 1: (a) A representative time step of PTV data discretized via Voronoi tessellation. A scalar field ($\Lambda$) has been measured for particles located at $[X_n, Y_n]$. Particles are enclosed by Voronoi cells ($\Pi$) with areas ($\Omega$). Empty and filled circles indicate unbounded and bounded particles, respectively. (b) Geometric features of cell $\Pi$ pertaining to particle $P$. $\Pi$ is neighboured by a set of particles enumerated by $I$, whose locations $\Lambda_I$, have been measured. The side lengths that $\Pi$ shares with cells enclosing particles $I$ are denoted as $l_I$ and the distances between $P$ and particles $I$ are denoted as $d_I$. The triangle enclosed by $l_I$ and $P$ is $A_I = \frac{1}{4}d_I l_I$. 

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Given the simplifications described above, Eq. 6 can be written in the following discretized form:

\[
\nabla (\Lambda) = 2 \sum_{I=1}^{J} (\Lambda_0 + \Lambda_I) \hat{n}_I l_I \sum_{I=1}^{J} (l_I d_I). 
\]

(7)

Note that Eq. 7 is only valid for two-dimensional flow fields and that modifications are required for application to three-dimensional flows. To determine a component of \( \nabla (\Lambda) \), \( \hat{n}_I \) is replaced by a dot product between \( \hat{n}_I \) and the component of interest. Thus, the \( x \) and \( y \) components of \( \nabla (\Lambda) \) are:

\[
\frac{\partial \Lambda}{\partial X_{t_0}} = 2 \sum_{I=1}^{J} (\Lambda_0 + \Lambda_I) (\hat{n}_I \cdot \hat{i}) l_I \sum_{I=1}^{J} (l_I d_I), \quad \frac{\partial \Lambda}{\partial Y_{t_0}} = 2 \sum_{I=1}^{J} (\Lambda_0 + \Lambda_I) (\hat{n}_I \cdot \hat{j}) l_I \sum_{I=1}^{J} (l_I d_I). 
\]

(8)

To calculate the spatial derivatives necessary to generate \( \nabla \Phi_{t_0+T} \), \( \Lambda \) may be replaced by \( X_{t_0+T} \) and \( Y_{t_0+T} \) since \( \Lambda \) represents a generic scalar field. It must also be noted that \( \nabla \Phi_{t_0+T} \) can only be calculated at bounded particles. The use of Eq. 7 is invalid and undefined for unbounded particles given their association with \( l_I \), which in this case are equal to infinity.

2.3 Non-Dimensional Parameters for Track Length Threshold

As described in the introduction, accurate LCS identification is dependent on two competing factors: the selected threshold track length and the number of particles that present with such a length. Two non-dimensional parameters are presented herein that are used throughout the current study to quantify both factors.

Consider a two-dimensional ubiquitous shear flow shown schematically in Fig. 2. A fluid flows over an obstruction (thick solid line) at a uniform velocity (\( U_0 \)). A resulting shear layer forms that divides the flow into an irrotational region and a recirculation region, hereafter referred to as the vortex. The vortex has a radius (\( R \)) and mean tangential velocity along its periphery (\( U_S \)). Superimposed onto the bulk two-dimensional flow are out-of-plane motions with a velocity scale (\( U_\perp \)). Two-dimensional Lagrangian particle tracking with a seeding density (\( d \) in m\(^{-2}\)) is performed within a laser sheet of thickness (\( l \)) to quantify the flow field. Particles with a temporal track length (\( T \)) are used to calculate the FTLE field, which increases the inter-particle spacing from approximately \( \sqrt{d}^{-1} \) to \( \varepsilon \), where \( \varepsilon \) is the inter-particle spacing between tracked particles. It is assumed here that \( U_S \sim U_0 \) and that the vortex is contained within a contour of length \( 2\pi R \). Furthermore, it is assumed that

\[
U_0 \sim U_\perp
\]

Figure 2: (a) Cross-sectional and (b) upstream views of a shear-layer flow downstream of a two-dimensional obstruction. Fluid travelling at a uniform velocity (\( U_0 \)) flows over an obstruction leading to the development of a recirculation region of radius (\( R \)) with a mean tangential velocity (\( U_S \)). Thick solid lines represent the obstruction, while thick dashed lines represent a solid surface or a line of symmetry. Two-dimensional Lagrangian particle-tracking data was acquired using a seeding density (\( d \)) and a laser sheet of thickness (\( l \)). Particles that are tracked with minimum length (\( T \)) and an inter-particle spacing (\( \varepsilon \)) are selected for LCS characterization. Superimposed onto the bulk two-dimensional flow are out-of-plane motions with a velocity scale (\( U_\perp \)). Velocities and distances are represented by single- and double-headed arrows, respectively.
$U_\perp$ is proportional to the root-mean-square of the out-of-plane fluctuations, which is denoted as $\sqrt{w'w'}$. From dimensional analysis, the mean error of the FTLE field ($E$) is a function of four $\Pi$ groups:

$$E = f(\Pi_1, \Pi_2, \Pi_3, \Pi_4) = f\left(\frac{1}{\sqrt{w'w'}T}, d\pi R^2, \frac{2\pi R}{TU_0}, \frac{\pi R^2}{\varepsilon^2}\right).$$

Each $\Pi$ group can be ascribed a physical meaning. $\Pi_1$, or the dropout ratio ($DR$), approximates the time required for a fluid parcel positioned at the centre of the laser sheet to move through the sheet that is subsequently non-dimensionalized by $T$. $\pi R^2$ represents the area of the vortex, while $d^{-1}$ approximates the mean area of particle-enclosing cells (Sec. 2.2) when threshold $T$ is not applied to the data. Thus, $\Pi_2$, referred to here as the seeding factor ($SF$), approximates the number of data nodes within the vortex when threshold $T$ is not applied. $\Pi_3$ and $\Pi_4$ are dimensionless representations of threshold $T$ and the number of particles that present with the specified track length, respectively. Specifically, $\Pi_3$ represents the number of times a particle located at the periphery circulates around the vortex and as such will be referred as the turnover number ($TN$). $\Pi_4$ approximates the area ratio between the mean particle cell and the vortex and will be referred to as the cell ratio ($CR$).

### 2.4 Algorithm Verification via Double-Gyre Flow

To verify the algorithm presented in Sec 2.2, the FTLE field of a data set with a known solution was calculated. Specifically, a double-gyre flow field consisting of confined counter-rotating vortices that represents a canonical test solution for LCS identification was generated using MATLAB R2011a (Mathworks, Natick, MA, USA) (Fig. 3). Between $D/2R = [0, 2] \times [0, 1]$ the flow is described by the stream-function:

$$\psi(\frac{X}{2R}, \frac{Y}{2R}) = \sin\left(\pi \frac{X}{2R}\right) \sin\left(\pi \frac{Y}{2R}\right)$$

where $R$ represents the horizontal span of each vortex. Here, the $X$ and $Y$ components of the vector field ($U$ and $V$, respectively) are defined as:

$$\frac{U}{2R} = \frac{\partial \psi}{\partial Y}, \quad \frac{V}{2R} = \frac{\partial \psi}{\partial X}.$$ 

The above described domain was discretized into $2.0 \times 10^6$ pixels and seeded with 10000 particles in random positions equating to a $d = 0.005$ particles-per-pixel (ppp) or equivalently, $d = 5000$ particles$/$(2$R$)$^2$. Particle track length was incrementally increased and the corresponding FTLE field was evaluated. As discussed previously, the number of tracks present at a certain threshold length will decrease as the threshold is increased. However, to evaluate the convergence of the code and understand the dependence of $E$ with $TN$, particle dropout was not permitted. This resulted in a $SF$ of 3900 and a constant $CR$ of 2800. Out-of-plane motion was not considered (i.e. $DR = 0$). The FTLE field was evaluated using $TN = 0.1$, $0.2$, $0.3$, $0.4$, $0.6$ and $0.8$. The resulting FTLE field and the sensitivity of the FTLE field as a function of increasing $TN$ are presented in Sec. 3.1.

![Figure 3: Velocity field of a steady-state double-gyre flow.](image-url)
2.5 Towed-Plate Experiment

Experiments were performed in a free-surface water channel with a mean width of 385mm and water depth of 470mm. A machined aluminium knife-edged plate with a span of 450mm and a root chord of \( c = 50 \text{mm} \) (aspect ratio = 9) was towed through quiescent water at an angle of 90°. The tow resulted in two equally sized vortices forming on the suction side of the plate. The kinematics of the tow consisted of an initial acceleration followed by a constant-velocity phase. The plate accelerated from rest to 0.1m/s at a constant acceleration of 0.1 m/s\(^2\), achieving its final velocity after a tow distance of a single chord length. The final tow velocity equates to a chord-based Reynolds number of 5000. From tow initiation to termination, the plate was towed a total distance of 13c. The tip-gap spacing between the plate and tunnel floor was maintained at < 5mm (0.1c) to mitigate free tip effects permitting the assumption of two-dimensional flow behaviour at the measurement plane located at the midspan of the plate.

Fig. 4 presents the experimental setup. Two-dimensional particle tracking data was acquired within a 1.2c field-of-view (FOV) positioned at the midspan of the plate. The axes origin of the FOV was positioned such that \( x = 0 \) and \( y = 0 \) coincided with the start position of the plate and the tip of the knife-edge, respectively. A 12.7mm diameter stainless steel sting was arranged on a motorized traverse such that the plate was vertically centred in the water channel during towing. The plate was towed in the x-direction through the FOV with the knife-edged tip centred vertically in the FOV. Towing was initialized at 0.25c such that the shadowing artifact caused by the plate was minimized. Images amenable to 2D-PTV analysis of the generated vortex were acquired using a Fastcam SA-4 camera (Photron, San Diego, CA, USA) at 125Hz using a full resolution of 1024 × 1024 pixels. Neutrally buoyant 100\( \mu \text{m} \) silver-coated, hollow-glass spheres (Potters Industries, Carlstadt, NJ, USA) with a Stokes number (Stk) of approximately 2.4 \times 10^{-3} were added to the quiescent water to serve as tracer particles. This is well below a Stk = 0.1 as recommended by Raffel et al. [14] to ensure tracer-accuracy errors of < 1%. The tracer particles were illuminated using a solid-state, 532nm continuous wave, 1W laser (Dragon Lasers, Changchun, Jilin, China) with a beam thickness of 6mm. Acquired images from 20 runs were imported to DaVis 8.1.3 (LaVision GmbH, Goettingen, Germany) for PTV analysis. The raw images were brightened by a fixed factor and particles identified using a threshold limit to achieve a recommended particle image density of approximately 0.005 ppp; see Cierpka et al. [15]. Lagrangian velocity data using a lower limit particle track length of two images were subsequently exported to MATLAB R2011a (Mathworks, Natick, MA, USA) for post-processing.

![Figure 4: (a) Schematic representation of the towed-plate experiment apparatus. Relevant equipment has been labelled. (b) Upstream view of the towed-plate experiment apparatus with relevant dimensions.](image-url)
To evaluate the $TN$ of the experiment, $U_0$ and $R$ were estimated as 0.1m/s and 0.25cm = 1.25mm, respectively. The FTLE fields were subsequently quantified at $TN = 0.1, 0.2, 0.3, 0.4, 0.6$ and 0.8 using a constant $DR$ and $SF$. The sensitivity of the mean FTLE field as a function of increasing $TN$ and $CR$ are discussed in Sec. 3.2.

3 Results & Discussion

The following section presents the FTLEs quantified from the simulated double-gyre and experimental towed-plate flow fields. Based on the quantified FTLEs, the fidelity of the proposed algorithm is reviewed. Furthermore, the implications of the dimensionless numbers presented in Eq. 9 on LCS resolution are also discussed. Finally, baseline values for both turnover number ($TN$) and cell ratio ($CR$) are suggested that would allow for adequate LCS extraction from the experimental data using the proposed unstructured algorithm.

3.1 Double-Gyre Flow

Fig. 5 presents double-gyre FTLE fields at $TN = 0.1, 0.2, 0.3, 0.4, 0.6$ and 0.8. The FTLE fields have been normalized by their respective maxima to allow for direct comparison. The central LCS ridge inherent of double-gyre flow is ambiguous at lower $TN = 0.1$ and 0.2 (Fig. 5a, b). The defining ridge becomes the dominant feature when $TN$ is increased to 0.3, after which the ridge becomes progressively more resolved with increasing $TN$ (Fig. 5c-f). The FTLE field generated by our algorithm at high $TN$ values matches that achieved by Shadden et al. [16], thereby demonstrating the fidelity of the proposed algorithm. The ridge displays high sensitivity to increases in $TN$ when $TN$ is low, where sensitivity is defined as the mean change in the quantified FTLE field upon increases in $TN$ of 0.1, and is evaluated using:

$$S(TN) = \frac{\sigma(TN) - \sigma(TN + 0.1)}{\sigma(TN)}.$$  (12)

![Figure 5: FTLE fields of a double-gyre flow using the algorithm across a range of turnover numbers ($TN$). The FTLE field is highly sensitive to increases in $TN$ at low values. As expected, changes in FTLE sensitivity become dampened upon use of greater $TN$ values.](image-url)
In contrast, the resulting change in sensitivity is dampened upon use of increasing $TN$ (i.e. $TN > 0.3$). Sensitivity as a function of $TN$ across $0.1 < TN < 0.7$ is plotted in Fig. 6. The relationship between sensitivity and $TN$ indicates convergence of the FTLE field, which is to be expected as $TN$ is increased. The FTLE field is highly sensitive to changes at lower $TN$; however, sensitivity was reduced with use of $TN > 0.3$, presenting with a value of approximately $0.2$. The results presented here indicate that quantification of the defining ridge associated with a double-gyre flow requires the use of a minimum $TN = 0.3$. Additionally, our results showed that FTLE sensitivity remained relatively constant with the use of greater $TN$ (i.e. $0.4$, $0.6$, $0.8$), suggesting only modest change in ridge resolution.

### 3.2 Towed-Plate Experiment

The purpose of the double-gyre simulation was two-fold: first to verify the ability of the proposed algorithm presented in Sec. 2.2 to quantify a known LCS solution; and secondly, to gain insight into the relationship between selected particle track length, represented here by $TN$, and subsequent LCS identification. The current section reinvestigates the effects of $TN$ on LCS resolution using experimental data of the resulting flow generated behind a towed knife-edged plate that in this instance generates particle dropout.

Fig. 7 presents the quantified FTLE fields of vortical flow generated by the towed plate. In each image, the plate was towed from left to right, generating a free shear layer that moves from right to left. The left, centre and right columns of Fig. 7 display the quantified FTLE fields about the plate at dimensionless times $t^* = (U_0 t)/c$ of $0.16$, $0.48$ and $0.8$, respectively. The figure is further organized such that the first, second, third and fourth rows present the resulting FTLE fields quantified at $TN = 0.1$, $0.2$, $0.3$ and $0.4$, respectively. At $t^* = 0.16$, the FTLE ridge loops around the plate, enclosing the tip irrespective of the $TN$ used. However, ridge connection to the plate, as well as the spacing between the ridge and the plate tip, progressively increase as a function of $TN$ (Fig. 7a, d, g, i). Furthermore, the characteristic ridge loop that encircles the plate tip at $t^* = 0.16$ is most prominent and well-defined at $TN = 0.2$ (Fig. 7d). However, the LCS ridge quantified at $t^* = 0.8$ using a $TN = 0.3$ appears more defined than that using $TN = 0.2$, which is obscured by its increased thickness and by the presence of spurious cells that exhibit high FTLE values (Fig. 7f, i).

In contrast to quantified FTLE fields using $TN = 0.2$ and $0.3$, the FTLE ridges generated using $TN = 0.1$ and $0.4$ present with decreased coherence. Specifically, a partial completed loop encircling the plate tip is resolved at $t^* = 0.16$ using $TN = 0.1$ (Fig. 7a). The LCS becomes progressively disorganized in accordance with the presence of discontinuities in ridge definition as $t^*$ is increased. The final quantified ridge at $t^* = 0.8$ presents a discontinuous LCS with a coherence greatly reduced from that quantified using $TN = 0.2$ and $0.3$ (Fig. 7b, c). Use of $TN = 0.4$ generates an FTLE ridge absent of the defined loop resolved using a lower $TN$ at $t^* = 0.16$ (Fig. 7h). The definition of the ridge is improved using $TN = 0.4$ at subsequent time steps to that using $TN = 0.1$; however, the ridge failed to present a well-defined connection to the plate as observed using $TN = 0.2$ and $0.3$. 

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Figure 7: FTLE fields of vortical flow generated by the towed plate. FTLE fields are plotted using turnover numbers \( (TN) \) of 0.1, 0.2, 0.3 and 0.4 at corresponding dimensionless times of \( t^* = tU_o/c = 0.16, 0.48 \) and 0.8.
The sensitivity ($S$) of the FTLE field from a towed plate as a function of turnover number ($TN$). The increase in sensitivity that occurs at $TN = 0.3$ suggests that the FTLE field proceeds to diverge when $TN > 0.3$. The divergence is the result of an insufficient cell ratio ($CR$).

Similar to the double-gyre flow field, $TN = 0.3$ produced a well-resolved and defined LCS ridge at all $t^*$ presented. However, application of $TN > 0.3$ to our experimental flow generated an increasingly disorganized and poorly defined LCS ridge. To quantify this reduction in ridge coherence, the sensitivity of the FTLE field as a function of $TN$ was calculated using Eq. 12. Fig. 8 plots three separate curves for sensitivity corresponding to the three presented $t^*$ of 0.16, 0.48 and 0.8. Unlike the double-gyre simulation, sensitivity of the experimentally-acquired FTLE fields did not continue to decrease with increasing $TN$. Instead, a general decrease in sensitivity is first observed, followed by a subsequent increase at $TN = 0.3$ and 0.4. The trend observed at all three $t^*$ suggests that the FTLE field initially converges with increasing $TN$, but proceeds to diverge upon use of $TN > 0.3$. At $t^* = 0.8$, it is noted that the decrease in sensitivity using $TN = 0.4$ following its increase at $TN = 0.3$ is not indicative of re-convergence. It was observed that ridge connection to the suction side of the plate was not resolved at $t^* = 0.8$ (see Fig. 7) using a $TN = 0.4$ in comparison to the prominent ridge connection quantified using $TN = 0.2$ and 0.3.

To understand the rationale behind the FTLE divergence at $TN > 0.3$, consider Fig. 9, which plots $CR$ as a function of $TN$. The relationship between $CR$ and $TN$ is an exponential decay of the form $Ae^{-Bx}$, which is represented by the black curve. The qualitative discussion in relation to Fig. 7 and the presented sensitivity of the FTLE fields in Fig. 8 suggests that the use of a $0.2 < TN < 0.3$ generates the most well-defined LCS ridge that characterizes our experimental flow conditions, corresponding to a $CR \sim 200$ (Fig. 9). This suggests that the benefits of using a greater $TN$ to resolve an LCS are diminished at $CR < 200$. Thus, given the results from the double-gyre and experimental vortical flows, it is suggested that $CR$ and $TN$ values of at least 200 and 0.3 must be achieved to adequately resolve an LCS from 2D-PTV data. These benchmark values are expected to be independent of both dropout ratio ($DR$) and seeding factor ($SF$), both of which would affect the shape of the black curve fit in Fig. 9.

4 Conclusions

The current study presents a novel algorithm to identify LCSs directly from unstructured data generated from 2D-PTV experiments. Opposed to traditional LCS quantification methods, the algorithm discretizes flow-map data at each time step via a Voronoi tesselation and subsequently calculates the spatial gradient using a discrete form of the divergence theorem without the requirement of fixed-grid interpolation. Through dimensional analysis, four $\Pi$ groups were presented as influencing factors on LCS identification. The groups quantified out-of-plane particle motion, particle seeding density, particle track length threshold and the number of particles that present with the track length threshold. In simulations and experiments described herein, the first two $\Pi$ groups, dropout ratio ($DR$) and seeding factor ($SF$), were held constant. The latter two, turnover number ($TN$) and cell ratio ($CR$), were manipulated to investigate their specific influence on LCS identification in both a synthetic flow with a known LCS solution and in experimentally-acquired data of two-dimensional free shear flow.
Steady double-gyre flow was simulated to both test the fidelity of the proposed algorithm and investigate the effect of $TN$ on LCS identification. Seeding density was set to 0.005 ppp, $CR$ was held constant at 2800 and out-of-plane motion was not considered. FTLE fields evaluated using $TN = 0.1, 0.2, 0.3, 0.4, 0.6$ and 0.8 demonstrated a convergence towards double-gyre FTLE fields presented by Shadden et al. [16]. Furthermore, the sensitivity of the FTLE field was found to be relatively constant with the use of a high $TN$. The agreement in double-gyre FTLE quantification with Shadden et al. [16] and the relationship between sensitivity and $TN$ served to verify the ability of the proposed algorithm to quantify an LCS with a known solution.

The algorithm was subsequently tested using experimental 2D-PTV data of nominally two-dimensional flow generated by a towed plate in a quiescent free-surface water channel. The plate with a chord of 50mm was accelerated at 0.1m/s$^2$ to a steady velocity of 0.1m/s, equating to a chord-based Reynolds number of 5000. FTLE fields were generated at $t^* = 0.16, 0.48$ and 0.8 using $TN = 0.1, 0.2, 0.3$ and 0.4. Distinct LCS ridges were observed using $TN = 0.2$ and 0.3 at all $t^*$ presented. Conversely, the FTLE field presented with a disorganized and discontinuous ridge using $TN = 0.1$ and 0.4. FTLE sensitivity as a function of $TN$ generally decreased up to the use of $0.2 < TN < 0.3$, after which an increase was observed. This suggests that the FTLE field initially converges with increasing $TN$ but proceeds to diverge upon use of $TN > 0.3$. The divergence observed using $TN > 0.3$ is attributed to a decrease in $CR$ that is insufficient to resolve the LCS ridge. This suggests that a compromise is necessary between track length and the density of tracked particles to resolve an LCS where an insufficient number of particles have long track lengths.

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