Effect of poor discretization in weighted cross-correlation for 3D PIV

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Abstract

The extension to the third dimension of the well-assessed PIV high accuracy cross-correlation based algorithm involves a consistent increase of the processing time, and several ideas have been developed to reduce it. The techniques avoiding redundant calculations in case of overlapped interrogation windows are so far the most effective; among the others, block cross-correlation (i.e. the contribution to the cross-correlation maps are computed on sub-volumes in pre-processing, and then summed) is appealing for its intrinsic simplicity. However, in this case weighting windows cannot be rigorously applied in the cross-correlation step, unless it is acceptable to replace them by a piecewise version. The degrading effects due to the poor discretization are thoroughly discussed by a theoretical assessment and virtual simulations. The results show that, in case of adoption of weighting windows and overlap lower than 87.5%, block correlation introduces unacceptable artifacts, and different solutions have to be adopted to accelerate the processing.

1. Introduction

Three-dimensional velocity measurement has been a great challenge for the experimentalists in the last decades. Tomographic Particle Image Velocimetry (Tomo-PIV) is a relatively recent (Elsinga et al 2006) three-dimensional three components (3D3C) anemometric technique. Tomo-PIV has shown its great potential in a wide range of applications, as wall turbulence in subsonic (Schroder et al 2011, Elsinga et al 2012) and supersonic regime (Elsinga et al 2010), as well as the study of the wake of bluff bodies (Scarano and Poelma, 2009), of jet flows (Violato and Scarano 2011), and so on. The outstanding features of the technique impose to focus the attention on its limit of applications and its margins of improvement.

The Tomo-PIV algorithms are structured in 2 main parts: tomographic reconstruction and 3D PIV. The new ring of the chain immediately captured the attention of the researchers, both to assess and improve the accuracy (Novara et al 2010). Furthermore, the computational cost of the tomographic reconstruction stimulated the development of more efficient algorithms (Atkinson and Soria 2009, Discetti and Astarita 2012a).

As for the extension to the 3D scenario of the well assessed 2D high accuracy PIV interrogation algorithms, the investigation has been very limited so far. The availability of the full velocity gradient tensor can lead to new exciting developments in this sense (Novara et al 2011); on the other hand, the algorithm developers have to be concerned with the noticeable increase of computational cost due to the introduction of the third dimension. The processing time to perform 3D cross-correlation with the aid of FFT is more than one order of magnitude higher than the time needed to reach the same task on 2D interrogation spots; in addition to this, the number of vectors to be computed can be consistently larger than in the case of 2D PIV, adding roughly another order of magnitude. For this reason some researchers (Discetti and Astarita 2012b, Ziskin and Adrian 2011) have proposed different solutions to reduce the computational cost. Ziskin and Adrian (2011) implemented a volume-segmentation tomographic technique, in which the 3D velocity field is obtained by using 2D cross-correlation on several planes along different directions. Discetti and Astarita (2012b) have investigated the application of multi-resolution interrogation, sparse matrices format and direct efficient cross-correlation, with redundant operations reduction by enjoying the overlap of the interrogation volumes. The main finding is that pre-calculating the contribution to the cross-correlation coefficients on blocks (Roth and Katz, 2001) is very efficient when the overlap ranges between 25% and 75% (up to 800 times acceleration for each iteration), while in case of highly overlapped windows the pre-calculation of the contribution on lines of...
voxxels is more effective. It is worth to underline that all these methods result in the same final accuracy when no weighting windows are applied in the process.

On the other hand, it has been proved that weighting windows can be very useful for a wide range of purposes: Nogueira et al (1999) introduced a weighting window to stabilize the interrogation process based on image deformation methods, and in subsequent papers (Nogueira et al 2001, Lecuona et al 2002) it is highlighted that scales smaller than the interrogation windows can be followed if correctly sampled by the particle motion; Astarita (2007) reviewed the performances of a wide range of weighting windows and developed a theoretical model to determine the Modulation Transfer Function (MTF) of the interrogation process when weighted dense predictor averaging is employed to stabilize the process; Astarita (2009) used the weighting windows feature to assess an adaptive PIV interrogation method; Novara et al (2011) made large use of spatially oriented Gaussian weighting kernels in the implementation of a 3D PIV adaptive interrogation algorithm.

However, the adoption of weighting windows in cross-correlation computation introduces an unacceptable increase of processing time (this aspect is widely explained in the next section), urgently imposing to use the efficient “redundancy-free” algorithms by Discetti and Astarita (2012a). The authors clearly observed that in most of the applications the overlap ranges between 50% and 75%, so that the temptation of using the very efficient block cross-correlation can be compelling. On the other hand, in this case the weighting windows cannot be applied rigorously since they have to be replaced by a piecewise version, since each block can be weighted by a single value.

The aim of the present work is to warn the experimenter that poor discretization of the weighting windows can lead to undesirable effects on the MTF of the process, even unexpected instability. In Sec. 2 the processing algorithm is presented and the theoretical model for the MTF estimation is introduced; in Sec. 3 the stability and the accuracy of the process is discussed using the theoretical model; eventually, the algorithm is tested on 3D simulated distributions of particles.

2. Cross-correlation algorithm with weighting windows

Following Nogueira et al. (1999), supposing cubic interrogation volumes without any loss of generality, the general formula for the cross-correlation coefficient is:

\[
\phi_{i,m,n} = \frac{\sum_{i,j,k} W_{i,j,k}^2 (A_{i,j,k} - \mu_A) (B_{i-l,j-m,k-n} - \mu_B)}{\sqrt{\sum_{i,j,k} W_{i,j,k}^2 (A_{i,j,k} - \mu_A)^2} \cdot \sqrt{\sum_{i,j,k} W_{i,j,k}^2 (B_{i-l,j-m,k-n} - \mu_B)^2}}
\]  

(1)

where \( A \) and \( B \) are the grey intensities of the interrogation volumes (IV) extracted from the first and the second reconstructed volume respectively, \( \mu_A \) and \( \mu_B \) are the mean intensities, \( W \) is the linear dimension of the IV, and \( w \) is a weighting window.

The steps of the reference algorithm that will be considered in this paper are shortly summarized in the following (more details can be found in Astarita and Cardone, 2005).

1. The predictor displacement field is calculated on a rather coarse grid using Eq. 1 with the aid of the FFT approach to compute the whole cross-correlation map, being the particles displacement initially unknown;
2. The predictor displacement field is interpolated on each pixel in order to deform the images accordingly;
3. A grid refinement may be executed in the evaluation of the displacement field. The IV size can be progressively reduced, since the image deformation compensates the in-plane displacement and the velocity gradients;
4. A corrector displacement field is evaluated on the deformed images. In the corrector estimation one could use a narrower search area of the peak (as already observed by Rohaly et al, 2002), e.g.
limiting the calculation only to the cross-correlation coefficients in the neighbourhood of the peak itself. As a borderline case, only the coefficients that are necessary to compute the gaussian interpolation of the peak can be evaluated;

5. The true displacement is obtained summing the corrector and a weighted average of the dense predictor over a prescribed window. Steps 2-5 are repeated till convergence. Discetti and Astarita (2012b) proposed to accelerate the predictor estimation by binning the distributions and using FFT, while the corrector displacement field is computed by direct correlation on a narrow search area. By indicating with $a$ the modulation associated with the step 1, with $b$ the modulation due to predictor averaging (step 5) and with $c$ the modulation due to interpolation (step 2), Astarita (2007) predicts the modulation at the generic $k$-th iteration as:

$$m^k = m^{k-1}bc + a(1-m^{k-1}c)$$  \hspace{1cm} (2)

Recalling that $m^0 = a$, Astarita (2007) introduces a stability criterion:

$$-1 < c(b-a) < 1$$  \hspace{1cm} (3)

Using Eq. 2-3 one can predict the MTF of the interrogation algorithm at the generic iterations, and the range of wavelengths causing instability (if any). In the following large use of these instruments will be made.

2.1 Block cross correlation

In common practice overlapping windows are used (overlap typically ranges between 50% and 75%). In this case, a relevant percentage of operations are repeated, and a strategy to reduce redundant calculations has to be assessed. Discetti and Astarita (2012b) propose 3 approaches to reach this goal, each one relying on the observation that the cross-correlation coefficients for each IV can be obtained by summing contributions of sub-volumes (Rohaly et al. 2002).

The most intuitive solution implies computation of the sums on blocks, which dimension could be set as the greatest common divisor between the IV linear dimension and the grid distance (this second parameter is replaced by the overlapping part of the IVs when the overlap is smaller than 50%). The idea is equal to that of Roth and Katz (2001), with the difference that this method is employed only to calculate a very limited number of coefficients, and no truncation is performed. Using this approach, as previously stated, the weighting function has to be replaced by a piecewise version, being each block contribution weighted with the same value. An example of block-version of the Blackman window is provided in figure 1 for different overlap percentages; in this case the choice is to weight each block with the value of the original window discretized with a number of points equal to the number of blocks.

![Fig. 1 Comparison between the standard Blackman weighting window and its piecewise version for several overlap values](nbl stands for n blocks).
Differently, one can pre-calculate sums along lines or planes (e.g. rows in the first case, or x-y planes in the second one). If sums along lines are calculated, the number of operations to be performed increases linearly with the number of vectors for each direction. With these last two approaches the weighting function can be correctly applied in the cross-correlation step.

3. Spatial resolution and stability of block weighting windows

In this section the MTF of block weighting windows is assessed. It will be assumed in the following that no modulation occurs due to interpolation (i.e. $c=1$); this condition is closely met in case of small grid distance and high accuracy interpolation scheme, e.g. B-Splines of high order, see Astarita (2008). In section 3.1 the modulation of solely the cross-correlation step will be considered, i.e. the modulation in case of the so called local approach, in which the corrector is directly summed to the predictor, without any spatial averaging of the dense predictor. In section 3.2 the stability of the algorithm in case of weighted predictor averaging will be considered.

3.1 Stability of the local approach

When no dense predictor averaging is performed (i.e. $b=1$, as proposed by Nogueira et al (1999)), and no modulation occurs due to interpolation ($c=1$), the Eq. 3 reduces to the requirement of positive $a$ for a stable process.

The MTF of a triangular, a Blackman, a Gaussian window ($\alpha=2$) and the weighting window proposed by Nogueira et al. (1999) are compared for different value of the overlap percentage, i.e. number of blocks (obviously, the case of 50% overlap cannot be considered, since in this case every weighting window is reduced to a top hat). The MTFs are plotted in figure 2 as a function of the normalized frequency, i.e. the ratio between the IV size $W$ (60 pixel for this investigation) and the wavelength $\lambda$.

The triangular function, and the weighting window proposed by Nogueira et al (1999), in their original versions, ensure the stability even with the local approach, since their MTF is positive definite; the Blackman and the Gaussian window, on the other hand, are to be coupled with some dense predictor averaging, due to slightly negative values for high frequency, mainly due to discretization effects. However, when the discretization is very poor, as in block cross-correlation, even the triangular function and the function proposed by Nogueira et al. (1999) present unexpected negative lobes in its MTF, causing instability in local approach application. The high frequency fluctuations are due to the abrupt variations of the weighting function between neighbouring blocks.

Otherwise stated, the poor discretized weighting functions can be considered as sum of properly weighted top hat windows, contributing with their own frequency response in building the MTF of the window. E.g., in case of 3 blocks discretization, the weighting window can be considered as a sum of a pedestal (see figure 1), with the height equal to that of the two external blocks, and a central top hat, one third of the total window wide. The MTF of the 3-blocks window is the sum of two aliased sinc functions: the first one, due to the pedestal, disappearing very quickly as the normalized frequency increases, the second one due to the small central kernel, introducing high frequency fluctuations.

In case of 3 and 4 blocks the performances are very similar, with wide oscillations and negative lobes: this is due to the fact that, when using an even number of blocks, the two central blocks are weighted with the same value (i.e., as a matter of fact, the window is composed by only $n/2$ blocks, the central one being twice as large as the other blocks). In all the reported cases using 80% overlap (i.e. 5 blocks) reduces the intensity of the fluctuations of the MTF, and determines negative lobes for wavelengths smaller than 5 times the IV size. Acceptably small fluctuations, however, are reached only when 7 blocks or more are used (i.e. about 86% overlap). Even in that case, the local approach is unstable; the stability of the process can be obtained by using weighted dense predictor averaging, i.e. $b\neq1$. 

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3.2 Stability of the process with weighted dense predictor averaging

Differently from the cross-correlation step, in step 5 the weighting windows can be applied rigorously (the computational cost of dense predictor summing is very small if compared to the other steps of the process, so that there is no reasonable incentive to use block summing). The modulation transfer function, in this case, can be obtained by using Eq. 2, where $a$ is the modulation due to piecewise weighting window, $b$ is the modulation of the original window with the proper discretization, and $c$ is again set to 1.

Recalling Eq. 3, in case of negative modulation in the cross-correlation step (some negative lobes have been observed in all the cases due to poor discretization) the process can be stabilized by a small positive or negative $b$. Different results can be obtained by using different windows in the dense predictor averaging; however, Astarita (2007) demonstrated that the main characteristics of the MTF are lead by the window chosen for the cross-correlation step. For this reason, from this moment on only the top hat window will be used for the dense predictor averaging, without any loss of generality.

In figure 3 the MTFs for the case of the Gaussian window with $\alpha=2$ after 3 and 100 iterations are reported. Three main discretization levels are investigated, i.e. 3, 4 and 5 blocks. The MTF of the original window is reported for comparison in figure 4.

Astarita (2007) observed that all the investigated weighting windows are stable when a predictor average
with a top hat window of size $W_b \geq 3$ is applied. This is not the case of poorly discretized weighting window. A summary of the minimum size of the predictor average window in order to have a stable process is reported in Table 1, however it is important to point out that this results may be consistently higher if a different weighting window is used to average the dense predictor. The process is always stable when the size of the dense predictor averaging window is more than 7 pixels wide when a top hat moving average is applied.

The MTFs of the block cross-correlation approach show already undesired features after only 3 iterations: whichever is the chosen size for the predictor averaging window, all the curves present a wide negative lobe for $W/\lambda \geq 3$ and $2 \leq W/\lambda \leq 4$ in the case of 3 and 4 blocks discretization, respectively. In the case of 5 blocks discretization the negative lobe affects only very small scales, and it is of moderate intensity.

![Fig. 3 MTF of a Gaussian window ($\alpha=2$) as a function of the normalized spatial frequency for different size $W_b$ of the window over which the predictor is averaged. Results reported after 3 iterations (a,b,c) and 100 iterations (d,e,f) in the case of 3, 4 and 5 blocks from the left to the right.](image1)

![Fig. 4 MTF of the original Gaussian window ($\alpha=2$) as a function of the normalized spatial frequency for different size $W_b$ of the window over which the predictor is averaged. Results reported after 3 iterations (a) and 100 iterations (b).](image2)
Increasing the number of iterations, the curves relative to $W_b=3$ and $W_b=5$ rapidly diverge for the 3 and 4 blocks discretizations, being the process unstable. In the case of 5 blocks discretization, even if the process is stable for $W_b=3$, the MTF has a negative peak of about -2.7 at $W/\lambda \approx 5.6$. However, one should note that $W_b=3$ is not advisable also for the original window, since the MTF presents wide oscillations, with a negative peak of approximately -1 at $W/\lambda \approx 1.9$.

When $W_b>20$, the MTF is substantially unchanged in all the cases, i.e. a good convergence is reached after few iterations. The case of moderately small $W_b$ (i.e. $W_b=10$) is slightly different. While for the original window the MTF is monotonically decreasing for small normalized frequency, and then slightly oscillating around zero with decreasing amplitude, in the case of 3 and 4 blocks discretization wide negative lobes are present, with non negligible intensity. The situation is much worse in the case of 4 blocks discretization, being the negative peak equal to about -0.8 at $W/\lambda \approx 1.8$. For the 3 blocks discretization the negative peak is much less intense (approximately -0.4). On the other hand, when 5 or more blocks are used and $W_b=10$, the region of negative MTF is practically the same of the cases with $W_b \geq 20$. This behavior is quite independent of the chosen original window.

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**Table 1** Minimum size (pixel) of the dense predictor averaging window $W_b$ in order to have a stable process.

### 4. 3D simulations

The performances of the process with the block weighted cross-correlation are assessed by using virtually generated distributions of particles. Two different layouts are investigated: zero-displacement on a 240 x 240 x 240 voxels volume, and one-dimensional sinusoidal displacement field on a 480 x 480 x 120 voxels (the displacement is along the $x$ direction, while the gradient is imposed on the $y$ direction, with wavelength variable between 600 and 40 voxels). Particles with Gaussian shape, 200 counts of peak intensity and $e^{-2}$ diameter of approximately 3 voxels are randomly distributed within the volumes; a Gaussian noise with a mean of 5 counts and a standard deviation of 2 counts is superimposed to the intensity distributions. In both cases the particle concentration is $6.5 \cdot 10^4$ particles per voxel, i.e. approximately 41.6 particles in an interrogation of 40 x 40 x 40 voxels. The large number of particles ensures that the investigated wavelengths in the second test layout are correctly sampled. The size of the interrogation window is 60 voxels (accordingly, in the case of the overlap of 66%, 75% and 80% the size of each block is 20, 15 and 12 voxels, respectively).

The accuracy is reported in terms of total error $\delta$:

$$\delta = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (u_i - u)^2}$$

(2)

where $N$ is the number of computed vectors, $u_i$ is the $i$-th measured vector, and $u$ is the correct displacement value. The first layout is particularly suitable to measure the accuracy of the process when a high accuracy interpolation scheme is used in step 2 of the algorithm (in the present work the velocity field is interpolated with a B-Spline of 10th degree), since the bias error is very small and it is not
influenced by the noise level (Astarita 2006); in this case the average total error is practically coincident with the error at zero displacement. As in Sec. 3, a Gaussian filtering window with parameter $\alpha=2$ is adopted in the cross-correlation, and a top hat moving window with variable size $W_b$ is used (2, 6, 10, 20, 30 and 60 voxels). The spatial resolution is assessed in terms of the MTF, estimated as in Astarita (2006):

$$MTF(\lambda) = 1 - \frac{\sum_{i=1}^{N}(u_i - u)^2}{\sum_{i=1}^{N}\left(\sin\left(\frac{2\pi y_i}{\lambda}\right)\right)^2}$$

(3)

4.1 Accuracy

In Fig. 5 the total error is reported as a function of the number of iterations to assess the convergence of the process (for the sake of clarity, the symbols are reported only each 5 iterations). In Fig. 5, left, the results in the case of 75% overlap (i.e. 4 blocks discretization) show that in the case of $W_b\geq 20$ the process converges quite rapidly (the average total error is substantially unchanged after 5 iterations). The cases with $W_b=2$ and 6 are within the range of instability of the method; as expected the error diverges, even if the divergence is quite slow in the second case (the modulation in the step 2 of the process due to the relatively large grid distance of 15 voxels can significantly restrict the instability range). In the case of $W_b=10$ the error is still increasing after 20 iterations, even if the rate of increase of the error is relatively low. In Fig. 5, right, the total error is reported for $W_b=6$ for the three different discretizations of (3, 4, 5 blocks) and for the exact discretization over the size of the window (in this last case, the process is theoretically stable, Astarita (2007)). The rate of increase of the error is consistently higher in the case of the process with block-discretized weighting windows in the cross-correlation step; the error is about 2 times higher with respect to the process with correct application of the weighting window after only 4 iterations, while it is more than 2.5 times higher after 15 iterations.

![Fig. 5](image)

**Fig. 5** Total error as a function for the number of iterations. Left: 4 blocks discretization (75% overlap) and different size of the dense predictor averaging window. Right: $W_b=6$ and variable overlap.

4.2 Spatial resolution

The MTF for the case of $W_b=6$ is reported in Fig. 6 for the different levels of discretization of the
weighting windows. One should note that the measured MTF is much lower than the theoretical estimate reported in Sec. 4. The main reason for this discrepancy is the sensitivity to the noise effects of the Eq. 3; the higher values of the theoretical MTF in the case of the 3-blocks discretization at higher frequency determine a stronger contamination of the MTF due to the noise (i.e. the measured MTF is consistently lower than the one relative to different levels of discretization). In addition, one should note that the theoretical estimated MTF does not take into account that the particles are randomly distributed, and they may not be able to properly sample the signal in the entire volume. However, this aspect is supposed to determine similar degrading effects to the MTF, regardless of the discretization level of the weighting window.

The total error, reported in Fig. 7 as a function of the wavelength, is determined both by the presence of noise and by the modulation of the velocity profile. This latter effect is dominating, and determines a higher error for the process with 3 blocks discretization, except for the smaller tested wavelength (in this case the MTF for the process with 3 blocks discretization is the highest one). In the case of 4 and 5 blocks discretization, the performances are very similar; however, since the results are evaluated after only 20 iterations, and the process with 4 blocks discretization in this configuration is unstable, the difference between the two methods can be asymptotically larger.

**Conclusion**

The effect of a poor discretization of the weighting windows in the scenario of fast algorithms to
compute the 3D cross-correlation is reviewed with the aid of theoretical models and virtual generated 3D distribution of particles. The results clearly underline that a poor discretization of the weighting window can determine a consistently different behavior in terms of impulsive response of the algorithm, determining instability even when it is not theoretically expected. The local approach is never recommended, unless the number of blocks used for the discretization is consistently high (at least 10 blocks, i.e. 90% overlap); on the other hand, in this case, the block cross-correlation has been proven to provide a lower speed-up with respect to other approaches (pre-calculation of the contribution to the cross-correlation coefficients along segments or planes, instead of blocks), in which the weighting windows can be applied correctly (Discetti and Astarita, 2012b).

The process with block-discretization requires large windows for the dense predictor average with respect to the case of the exact discretization of the weighting window over the whole size of the interrogation volume.

The virtual simulations highlight that the poor discretization of the weighting window determines a higher sensitivity to random noise, i.e. a larger total error. This is due to the strong fluctuation of the MTF at high frequency, generated by the abrupt variations within the weighting window. In summary, in the application of block weighted cross-correlation two rules of thumbs can be proposed:

- When possible, use a odd number of blocks: this is because, when using a even number of blocks, the central blocks are equally weighted, determining in the frequency response a larger negative MTF for frequencies ranging between \(0.5 \cdot N_{\text{blocks}} \leq W/\lambda \leq N_{\text{blocks}}\) (the two central blocks of the weighting function can be considered a top hat filter with size equal to twice the size of each block);
- Use wide dense predictor averaging windows (for the top hat approach \(W_b \geq 10\) is suggested when the discretization is performed on 5 or more blocks).

References


Discetti and Astarita (2012b) Fast 3D PIV with direct sparse cross-correlations (submitted to Exp Fluids)


