Third-order double-frame digital particle image velocimetry

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Abstract This article derives a method for correcting first-order or second-order shift-vector fields to achieve third-order accuracy. The main idea is to identify the most likely streamline with constant curvature from the first-order shift vector and its gradient. The work establishes a theoretical framework including the systematic errors of the first-order and second-order shift vector's absolute value and angle. Synthetic images of a stationary vortex are used to validate the proposed method. The approach is very general and can be applied to any first-order vector field achieved from window-correlation PIV, single-pixel ensemble-correlation PIV, particle tracking velocimetry or optical flow methods. It also works for all 3D extensions of the techniques, such as 3D-PTV or tomographic PIV.

1. Introduction

Digital particle image velocimetry (DPIV) is a non-intrusive measurement technique that estimates the velocity field in a plane, or even in a volume, by measuring the displacement of particles in a certain time interval Δt. Due to the nature of the recording principle, each measured velocity vector represents a volume-averaged mean motion of the discretized and quantized tracer particle's diffraction images, rather than the actual velocity of the flow (Adrian and Westerweel, 2010; Raffel et al, 2007). To maximize the information output of DPIV measurements the dynamic spatial range (DSR) and the dynamic velocity range (DVR) should be maximized (Adrian, 1997).


To achieve a large DVR as well as to get accurate estimations of the velocity and quantities derived from it, it is necessary, especially for low magnifications, to maximize the particle image shift by selecting a sufficiently large time delay between each two illuminations. On the other hand, large time separation may cause a bias error in areas with curved streamlines, as illustrated in Fig. 1.

Figure 1 Particle path estimated from the position of the particle image at time t and t + Δt.

This work, however, focuses on a different approach: The curvature of the path will be estimated from the first-order shift vector field. The assumption that neighboring stream lines do not cross allows for the reconstruction of streamlines by the local shift vectors and their gradients. The article is divided into two sections. Section 2 discusses the third-order correction from an analytical point of view. Third-order accuracy is obtained from the first-order shift vector field and its gradients. In Sec. 3, the developed method is applied to synthetic DPIV data in order to verify the expectations.

2. Mathematical description

The evaluation of double-pulse double-frame DPIV images approximates the path of motion by a straight line. Starting from the first-order shift vector field, which connects the start point and the end point of this line segments, a third-order shift vector field is constructed in this section. The straight line is the shortest possible path, thus, for complex flows, it is likely that the actual path is longer. Hence, the absolute value of the estimated shift vector is generally underestimated. Assuming an actual path with constant curvature, as shown in Fig. 2, this bias error depends on the radius of the curvature $R$ and on the arc's angle $\xi$ (See Fig. 2).

In order to compensate for first and second order bias errors the parameters $\xi$ and $R$ must be extracted from the shift-vector field. Neighboring shift vectors that connect two points on the same streamline can be used to estimate the flow path's curvature, as illustrated in Fig. 3. The aim is to find a vector $\delta_1$ on a small radius $r \to 0$ that fulfills the following condition:

$$\beta_1 + \varphi = \beta_2 - \varphi$$

Under this condition, all four points (start point and end point of shift vector and neighboring vector) fall on the same circle which represents a constant-curvature approximation of the path of motion. The starting point of the neighboring vector must be close to the starting point of the shift vector in order to align the vector tangential to the circle. Due to the finite grid spacing of DPIV.

Figure 2: First-order shift vector and third-order shift vector.

Figure 3: Third-order reconstruction by using neighboring vectors that follow the same path.
data points it seems useful to use a local (first-order) series expansion for the neighboring vector:

\[ \Delta X^{(i)}(X_0, Y_0) + \delta_i = \delta_i + \Delta X^{(i)}(X_0, Y_0) + \frac{\partial \Delta X^{(i)}}{\partial \delta_i} \cdot \delta_i \]  

(2)

in order to interpolate between the discrete data points. For a neighboring shift vector that fulfills the condition in Eq. (2) the first order shift vector \( \Delta X^{(i)}(X_0, Y_0) \) and a neighboring vector, located at \( (X_0, Y_0) + \delta_1 \), differ by the vector \( \delta_2 \):

\[ \Delta X^{(i)}(X_0, Y_0) + \delta_2 = \delta_1 + \Delta X^{(i)}((X_0, Y_0) + \delta_1) \]  

(3)

Under this condition, the desired angle \( \beta_1 \) of the tangent can be computed from the shift vector and its gradient as follows:

\[ \beta_1 = \beta_2 - 2\varphi = \arctan \left( \frac{r \cdot \sin \beta_1 + V_x \cdot \cos \beta_1 + V_y \cdot \sin \beta_1}{r \cdot \cos \beta_1 + U_x \cdot \cos \beta_1 + U_y \cdot \sin \beta_1} \right) - 2 \cdot \arctan \frac{V_y}{U_y} \]  

(4)

Where \((U_0, V_0)\) are the components of the first-order shift vector and \(U_x, U_y, V_x,\) and \(V_y\) are the partial derivatives with respect to \(X\) and \(Y\), respectively. The angle \( \varphi \) is the orientation of the first-order shift vector. It is important to note that \( \beta_1 \) is not given explicitly in Eq. (4). However, it is possible to compute \( \beta_2 \) for a set of \( \beta_1 \) (ranging from \(-\pi/2\) to \(+\pi/2\)) and find the solution numerically. Once the angle \( \beta_1 \) is found, the radius \( R \) of the circle can be computed:

\[ R = \frac{\left| \Delta X^{(i)}(X_0, Y_0) \right|}{2 \cdot \sin \left( \frac{\xi}{2} \right)} \]  

(5)

Where the arc's angle is given by:

\[ \frac{\xi}{2} = \beta_1 \]  

(6)

Finally, from the radius and the arc's angle, the third-order shift vector is computed as follows:

\[ \Delta X^{(3)}(X_0, Y_0) = \left\{ \begin{array}{ll}
\frac{\Delta X^{(i)}(X_0, Y_0)}{\sin \beta_1} \cdot \beta_1 \cdot \cos(\beta_1 + \varphi) \\
\frac{\Delta X^{(i)}(X_0, Y_0)}{\sin \beta_1} \cdot \beta_1 \cdot \sin(\beta_1 + \varphi)
\end{array} \right. \]  

(7)

The absolute value of the third-order shift-vector from Eq. (7) is enlarged from the tangent’s length to the arc length and the angle is rotated such that the vector is aligned tangential to the approximated circle.

For 3D PTV or tomographic PIV the approach works in the same way: A neighboring shift vector that fulfills Eq. (1) allows for the third-order reconstruction. Additionally, the vectors \( \delta_1 \) and \( \delta_2 \) need to be computed for the third-order shift vector.
\(\delta_2\) from Fig. 3 must lie in the same plane in order to estimate the most likely path of motion with constant curvature.

3. Synthetic example: Lamb-Oseen vortex

The Lamb-Oseen vortex is a frequently used vortex model in fluid dynamics. The circumferential velocity component \(V_\phi\) of this vortex model is given by the following equation:

\[
V_\phi(r) = \frac{\Gamma}{2\pi r} \left( 1 - \exp \left( -\frac{r^2}{r_c^2} \right) \right)
\]

(7)

Where \(\Gamma\) is the total circulation and \(r_c\) the vortex core radius. The radial velocity of a Lamb-Oseen vortex is zero, thus the streamlines have a constant curvature. Therefore, this vortex is well suited to test the capability of the developed method. The shift vectors are (in theory) estimated with third order accuracy so no bias error should be visible after correction. To demonstrate the ability of the developed method, synthetic images are generated, as described in Sec. 3.1, and evaluated using different methods: window-correlation for a single pair of DPIV images (Sec. 3.2), ensemble-averaged window-correlation (Sec. 3.3) as well as single-pixel ensemble-correlation (Sec. 3.4).

3.1 Image generation

The analysis based on synthetic images gives full control of all simulation parameters and is thus perfectly suited to detect sensitivities and to deduce what is relevant and what is not (Kähler et al., 2012). 10,000 synthetic image pairs, 512×512 px in size, with a stationary Lamb-Oseen vortex in the image center were generated for different total circulations \(\Gamma\).

A particle image density of 25% was applied, meaning that 25% of the image area was covered by particle images. The digital particle image diameter was \(D = 3\) px, hence, the number of particles per pixel was \(N_{ppp} = 0.035\) on average. A Gaussian particle image shape was assumed:

\[
I(X,Y) = I_0 \exp \left[ -8 \cdot \left( \frac{(X - X_0)^2}{D} + \frac{(Y - Y_0)^2}{D} \right) \right]
\]

(8)

Where \((X_0,Y_0)\) is the randomly chosen center position and \(D\) is the diameter at \(1/e^2\) of the maximum intensity (4 times the standard deviation). The maximum intensity of the particle images was \(I_0 = 2^{14}\). From Eq. (8) the discrete pixel's gray values were computed from the integral over the pixel's area, corresponding to a sensor fill-factor of 1. Non uniform pixel response, as outlined in Kähler (2004) was not considered.

The intensity distribution of the first synthetic DPIV images is given by following equation:

\[
A(X,Y) = \sum_{p=1}^{P} \int_{X-0.5}^{X+0.5} \int_{Y-0.5}^{Y+0.5} I_p(X,Y) dx dy
\]

(9)

Where \(P\) and \(p\) are the number of particle images and the corresponding control variable, respectively. On top of the particle image intensities, a Gaussian noise with zero mean and a standard deviation of \(1\% \times I_0\) was added. Finally, the intensity distribution was converted into a 16-bit unsigned integer matrix.

For the second images the center positions of the particle images were shifted such that the
distances to the vortex center remained constant and the arc lengths are proportional to the circumferential velocity from Eq. (7). The image generation was performed using MatLab functions.

### 3.2 Window-correlation

Instantaneous velocity fields are often of interest since they can visualize the shape of turbulent structures, which helps to understand the physical nature of turbulent flows and allows for developing models (Adrian et al, 2000; Robinson, 1991; Herpin et al, 2008). Furthermore, instantaneous vector fields, captured at sufficiently high frequency, give the possibility to analyze the temporal development of flow structures or to find characteristic frequencies.

To demonstrate the benefit of third-order correction, single DPIV image pairs of a synthetic vortex are processed. Figure 4 shows an instantaneous vector field computed with window-correlation using DaVis8.1 (by LaVision GmbH), including multi-pass evaluation with decreasing window size, iterative window shifting, window deformation, and Gaussian window weighting. The background color in Fig 3.1 represents the absolute value of the third-order shift vector. Shift vectors with first, second, and third order accuracy are shown in different colors for X = -4 px. The vortex core radius and the total circulation were $r_c = 20$ px and $\Gamma = 104$ px$^2$, respectively. This results in a relatively large maximum particle image shift of 36 px. Such a large distance is required for accurate measurements with high DVR, as for example in Hain et al (2009). Figure 5 shows the circumferential shift vector component for three different circulations ($\Gamma = [1,000; 5,000; 10^4]$ px$^2$) estimated with first, second and third order accuracy using $8 \times 8$ px and $16 \times 16$ px interrogation windows.

It can be concluded from Figs. 4 and 5 that:

- for a maximum shift vector length of 3.6 px, the maximum bias error of first-order and second-order evaluation is 0.01 px which is in the order of (or even below) the random error
- for regions with large gradients ($r < 30$ px), small interrogation windows are required for reliable velocity estimations
- the third-order correction results in significantly decreased bias for $8 \times 8$ px interrogation windows

Figure 4: Instantaneous third order shift vector field of a simulated Lamb-Oseen vortex using window-correlation with $16 \times 16$ px interrogation-window size. The first- (blue) second- (green) and third-order (black) shift vectors are shown for a cross section.

Figure 5: Estimated circumferential velocity component of a simulated Lamb-Oseen vortex using window-correlation with first, second, and third accuracy.
3.3 Averaged window-correlation

While instantaneous velocity fields are important for a deeper understanding of the flow physics, they are usually useless for the comparison of different experiments or for the validation of numerical methods. Statistical values, such as the mean velocity and Reynolds stresses are better suited for this purpose.

To demonstrate the benefit of third-order correction for sum-\(\sigma\) -correlation evaluation, 100 DPIV image pairs of a synthetic vortex were processed. Figure 6 shows an averaged vector field computed with window-correlation using DaVis8.1 (by LaVision GmbH), including multi-pass evaluation with decreasing window size, iterative window shifting, window deformation, and Gaussian window weighting. As before, the vortex core radius and the total circulation were \(r_c = 20\) px and \(\Gamma = 104\) px\(^2\), respectively. Figure 7 shows the circumferential shift vector component for three different circulations estimated with first, second and third order accuracy using \(8 \times 8\) px interrogation windows.

As before, Figs. 6 and 7 show that the third-order correction reduces the bias error significantly. However, for regions with large gradients (\(r < 30\) px) a small systematic deviation remains even for third-order correction. This bias is caused by the final size of the interrogation windows.

Figure 6: Averaged third order shift vector field of a simulated Lamb-Oseen vortex using window-correlation with \(8 \times 8\) px interrogation-window size. The first- (blue) second- (green) and third-order (black) shift vectors are shown for a cross section.

Figure 7: Estimated circumferential velocity component of a simulated Lamb-Oseen vortex using window-correlation with first, second, and third accuracy.

3.4 Single-pixel ensemble-correlation

Single-pixel evaluation can be used for a large amount of DPIV image pairs and results in improved spatial resolution (Kähler et al, 2012). This is of importance for the analysis of flow fields with strong gradients. Single-pixel ensemble-correlation was first applied by Westerweel et al (2004) for stationary laminar flows in microfluidics. In the last years, the approach was extended for the analysis of periodic laminar flows (Billy et al, 2004), of macroscopic laminar, transitional and turbulent flows (Kähler et al, 2006) and for compressible flows at large Mach numbers (Kähler and Scholz, 2006; Bitter et al, 2011). Scholz and Kähler (2006) have extended the high-resolution evaluation concept also for stereoscopic PIV recording configurations. Recently, based on the work of (Kähler et al, 2006), the single-pixel evaluation was further expanded to estimate Reynolds stresses in turbulent flows with very high resolution (Scharnowski et al, 2012).
For single-pixel ensemble-correlation, the correlation functions $C(\xi,\psi,X,Y)$ can be computed from a pair of PIV images $A(X,Y)$ and $B(X,Y)$ as follows:

$$
C(\xi,\psi,X,Y) = \sum_{n=1}^{N} \left[ A_n(X,Y) - \bar{A}(X,Y) \right] \left[ B_n(X + \xi, Y + \psi) - \bar{B}(X + \xi, Y + \psi) \right] / \sigma A(X,Y) \cdot \sigma B(X + \xi, Y + \psi)
$$

(10)

where the standard deviation is given by:

$$
\sigma A(X,Y) = \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} \left[ A_n(X,Y) - \bar{A}(X,Y) \right]^2}
$$

(10)

$(X,Y)$ are discrete coordinates of the pixel in question in both images and $(\xi,\psi)$ are the coordinates on the correlation plane. The maximum position of the correlation function $C$ represents the mean shift vector with first-order accuracy. Second-order accuracy can be achieved by moving the first-order vectors to their middle position as demonstrated in Scholz and Kähler (2006). Furthermore, the third-order correction discussed in Sec. 2 can be directly applied to the first-order vector field. In the case of laminar flow, the third-order correction estimates the most likely streamline with constant curvature. On the other hand, for turbulent flows, the third-order correction estimates the mean path with constant curvature which differs from the actual streamline for instantaneous vector fields, in general.

In order to verify the capability of the suggested third-order correction, 10,000 synthetic DPIV images were generated and analyzed. Figure 8 shows an averaged vector field computed with single-pixel ensemble-correlation using Eq. (10). As before, the vortex core radius and the total circulation were $r_c = 20$ px and $\Gamma = 104$ px$^2$, respectively. Figure 9 shows the circumferential shift vector component for three different calculations estimated with first, second, and third order accuracy. Figure 9 shows that the third-order correction causes a reduced bias error also in the case of single-pixel evaluation. Furthermore, even for regions with large gradients ($r < 30$ px) the third-order accurate data points fit the theoretical curve nicely. Thus, the remaining bias of the window-correlation approach could be eliminated.

Figure 8: Averaged third order shift vector field of a simulated Lamb-Oseen vortex using single-pixel ensemble-correlation. The first- (blue) second- (green) and third-order (black) shift vectors are shown for a cross section.

Figure 9: Estimated circumferential velocity component of a simulated Lamb-Oseen vortex using single-pixel ensemble-correlation with first, second, and third accuracy.
Figure 10 shows the difference of the shift vector’s absolute value (left) and angle (right) for first-order (top) and second-order (bottom) accuracy with respect to the third-order shift vector. The simple first order approximation underestimates the circumferential velocity (Fig. 10a) and causes a strong inward-facing radial component, indicated by the positive sign of the angular bias in Fig. 10b. The second-order approximation gives the right values for the shift vector’s angle (Fig. 10d) but the circumferential component is still biased. Figure 10c shows that the velocity in the core (r < 20 px) is overestimated, while the velocity outside the core is underestimated. The third-order accurate estimation does not suffer from any bias error for the tested Lamb-Oseen vortex. These findings are in agreement with the theoretical results of Sec. 2.

![Figure 10 Bias error of the shift vector's absolute value (left) and angle (right) for first-order (top) and second-order (bottom) accuracy using single-pixel ensemble-correlation.](image)

**Conclusion**

The presented method allows for the estimation of 2-D velocity fields from DPIV data with third-order accuracy. This result is of great importance for the accurate estimation of velocity fields from DPIV data acquired at low optical magnification, as required for high dynamic spatial range. The developed method was validated by analyzing synthetic DPIV images of a stationary Lamb-Oseen vortex. The estimated third-order vector fields demonstrated the benefit for standard window-correlation (instantaneous and ensemble-averaged) as well as for single-pixel ensemble-correlation.
Due to the third-order correction, the bias error of the shift vector's absolute value and angle is significantly reduced.

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