Measurement of turbulent spatial and temporal length scales is an increasingly important aspect of analysis in fluid mechanics. This information is derived either from the cross correlation or the cross spectrum between two measurements. The use of laser Doppler velocimetry (LDV) further complicates the determination of these properties because of the random data acquisition associated with LDV measurements. The calculation of the cross spectrum obtained for two LDV systems using sample-and-hold reconstruction procedures is considered. The errors in cross spectrum estimates together with coherence function calculations (i.e. frequency domain cross covariance) are derived for correcting the cross spectrum and, hence, the phase and the coherence are proposed. The results from numerical simulations are shown to produce good results up to and beyond the equivalent Nyquist frequency based on the mean data rate for the sample-and-hold method. Using simulated data with prescribed coherence and phase, a comparison of the results with those obtained from the correlation slotting method is presented and the advantages and limitations of each technique is discussed.
1. Introduction

For many applications in fluid mechanics, fundamental understanding of turbulence related phenomena depends not only on the correct estimation of the turbulence frequency spectrum but also on accurate estimation of the spatial correlation of the turbulence and on the degree of correlation of the turbulence with other dynamic parameters. This is usually expressed as a normalised cross correlation or cross covariance function but the use of the coherence function (normalised cross spectrum) and phase provides considerably more information as they are frequency dependent measures of interaction. In aeroacoustics, the prediction of noise generated for boundary layer flows and jet flows is dependent on the cross spectrum of turbulence (e.g. Durant et al., 2000). Thus, for many measurements, not only is the auto spectrum or PSD required but the cross spectrum is also to be determined. Two point correlation measurements using two component LDV systems have been reported by Erickson & Karlson (1995) who defined the measurement volume requirements with respect to the Taylor microscale in order to obtain accurate correlation estimates at small spacings. Trimis & Melling (1995) have reported an improved method for two point space correlations and suggest the possible use of slot correlation of velocity pairs to obtain the space-time cross correlation. For both of these applications, a classical space correlation approach was used for the estimation of the cross covariance and the main problem was one of coincident measurement. Van Maanen et al. (1999) combined a fuzzy slotting technique with the local normalisation approach of van Maanen & Tummers (1996) to reduce the high variance of the estimate of the cross correlation. For two channel non-coincidence LDV data, Muller et al. (1998) proposed a procedure based on sample-and-hold reconstruction and compared the results from their method to those from the slot correlation technique.

The errors associated with the sample-and-hold process have been detailed by Boyer & Searby (1986), and Adrain & Yao (1987), and have been shown to comprise of a step noise which is white and adds a constant bias to the estimated spectrum and a low pass filter effect. Procedures by which these effects can be corrected and an estimate of the true spectrum obtained were proposed by Nobach et al. (1998) and further refined by Simon & Fitzpatrick (2004). For the latter, a discrete low pass filter was used together with an estimate of the noise to yield a corrected auto-spectrum.

In this paper, the sample-and-hold method of Simon & Fitzpatrick (2004) is extended to the determination of cross power spectra from non-coincident LDV data. The objective is to obtain the auto and cross spectra for two measurements and from these determine information on their inter relationship using coherence and phase as these give frequency dependent information on convective velocities and length scales of the turbulence. A series of numerical simulations are used to examine the procedures and the results are compared with those obtained from the using fuzzy slotting method with local normalisation as proposed by Van Maanen et al. (1999).

2. Sample-and-Hold Reconstruction

There are two conditions under which cross spectra are to be found from two LDV measurements, the first is where the data is coincident and the second is for the non-coincident mode. The errors arising when analysing sample-and-hold reconstructed data from two LDV processors operating in a non-coincident mode are considered here. A schematic of the system is shown in figure 1 where \( r_1(t) \) & \( r_2(t) \) are the sample-and hold reconstructed signals given in the frequency domain by:

\[
R_1(f) = L_1(f)\{U_1(f) + S_1(f)\} \]
\[
R_2(f) = L_2(f)\{U_2(f) + S_2(f)\} \]

where \( U_1(f) \) & \( U_2(f) \) are the Fourier transforms of the two velocity signals, \( u_1(t) \) & \( u_2(t) \). \( L_1(f) \) & \( L_2(f) \) are the low pass filters and and \( S_1(f) \) & \( S_2(f) \) are the Fourier transforms of the “step” noise associated with the sample-and-hold reconstruction as detailed by Simon & Fitzpatrick (2004). In the first instance, the reconstructed data is corrected to eliminate the low pass filter effects from both data sets so that

\[
X_1(f) = \frac{R_1(f)}{L_1(f)} \]
\[
X_2(f) = \frac{R_2(f)}{L_2(f)} \]

The cross spectrum of these is

\[
G_{12}(f) = \langle X_1^*(f)X_2(f) \rangle \\
= \langle \{U_1^*(f) + S_1^*(f)\} \{U_2(f) + S_2(f)\} \rangle 
\]
If it is assumed that the step noise contaminations \( s_1(t) \) & \( s_2(t) \) are un-correlated with each other and with both \( u_1(t) \) & \( u_2(t) \), this then becomes

\[
G_{u_1u_2}(f) = G_{s_1s_2}(f) \tag{1}
\]

Thus, the cross spectrum from the low pass filter corrected data is the actual velocity cross spectrum which can be inverse Fourier transformed to obtain the cross correlation function. More interestingly, the phase of \( u_1(t) \) & \( u_2(t) \) can be determined directly from the low pass corrected data and from this the convection speeds can be obtained as a function of frequency. Also, the spatial coherence is of significant interest as the length and time scales can be determined as a function of frequency from this. The coherence function between \( s_1(t) \) and \( s_2(t) \) can be written as:

\[
\{\gamma_{s_1s_2}(f)\}^2 = \{ G_{s_1s_2}(f) \}^2 / (\{G_{u_1u_2}(f)\}^2 + \{G_{u_2u_2}(f)\}^2)
\]

This will be negatively biased by the step noise contamination in both data sets and this bias will be greatest at high frequencies where the signal to noise ratio will be low. The actual coherence function is then be estimated from

\[
\{\gamma_{u_1u_2}(f)\}^2 = \{\gamma_{s_1s_2}(f)\}^2 \cdot [(1 + \alpha_1(f))(1 + \alpha_2(f))]
\]

where \( \alpha_1(f) = G_{s_1s_1}(f)/G_{u_1u_2}(f) \) and \( \alpha_2(f) = G_{s_2s_2}(f)/G_{u_2u_2}(f) \) are the estimates of the noise to signal ratios.

3. Correlation Slotting

Spectral estimation via slot correlation involves calculating all possible combinations of cross-product between the data points of two signals, which, plotted as a function of the associated time lags give an estimation of the cross-correlation function (CCF). The cross-products are then accumulated and averaged in equispaced bins (slots) in the correlation domain, giving a regularly discretised estimation of the autocorrelation function which can be subsequently windowed and Fourier transformed to give an estimate of the power spectral density of the signal.

The high variance associated with this approach can be mitigated by applying a triangular windowing function to each individual slot (fuzzy slotting), which allows cross-products from adjacent slots to contribute to the local estimate, and to normalise each slot using only those data points which contribute to that estimate (local normalisation). The technique was proposed by van Maanen (1999) with the fuzzy slotting technique of Nobach et al. (1998) combined with a local normalisation approach suggested by van Maanen & Tummers (1996).

\[
R_k = \left[ \sum_{j=1}^{N} \sum_{\ell=1}^{N} u_j u_\ell b_k(t_j - t_\ell) \right] \times \left[ \sum_{j=1}^{N} \sum_{\ell=1}^{N} v_j v_\ell b_k(t_j - t_\ell) \right]^{-1/2}
\]

where \( R_k \) is the ACF estimate for slot \( k \), \( N \) is the number of data points in the signal, \( b_k \) is the triangular windowing function used to perform the "fuzzy" operation, defined as

\[
b_k(t_j - t_\ell) = \begin{cases} 
1 & |t_j - t_\ell| \leq \Delta \tau - k \\
0 & \text{otherwise}
\end{cases}
\]

where \( \Delta \tau \) is the slot width. A Hanning window is then applied to the CCF whence via a Fourier Cosine transform the cross spectra can be obtained.

4. Simulations

A series of simulations were performed using a Kolmogorov type spectrum, \( G(f) \), to represent the spectra for the two measurements. From this, the time domain data for \( U_i(t) \) was computed using the inverse Fourier transform method as

\[
U_i(t) = \text{IFT} \left[ G(f) e^{i\theta(f)} \right]
\]

where \( \theta(f) \) is a random phase with a uniform distribution from 0-2\pi.
The time domain data for \( U_2(t) \) is then obtained by fixing the degree of coherence, \( \gamma^2(f) \), and the phase \( \phi(f) \), between the two data sets giving

\[
U_2(t) = \text{IFT} \{ \gamma(f).G(f)^{1/2}e^{-j[\phi(f)-\phi(f)']} \} + \text{IFT} \{ (1-\gamma^2(f))^{1/2}.G(f)^{1/2}e^{j\psi(f)} \}
\]

where \( \psi(f) \) is an independent random phase with a uniform distribution from 0 to \( 2\pi \). In this way, it is possible to simulate the effect of decay and convection as is observed in turbulent flow measurements.

Although it is possible to vary the spatial coherence of the turbulence as a function of frequency, in order to examine the efficacy of the proposed procedures, data sets were generated for constant coherences of 1.0, 0.8, 0.5 & 0.3 with phases determined by a fixed time delay, \( \tau \), for each coherence so that the phase for each was determined from \( \phi=2\pi\tau \). Both signals were then sampled using independent Poisson distributed time intervals with selected mean data rates to represent the LDV measurements. The reference data was \( U_1(t) \) and this was sampled at a mean data rate of 500Hz. The second data set was resampled at 250Hz, 500Hz and 1000Hz to examine the effect of different mean data rates on the results. Data sets of 100,000 points with a time step of 2.5\( \mu \)sec were used. Auto and cross spectra together with phase and coherence were then determined using both the sample-and-hold reconstruction method proposed and the results compared with those obtained from the correlation slotting method. For the sample-and-hold reconstruction, the data was resampled at 20kHz and the procedures described in section 2 implemented to determine the coherence and phase for each case. For the slotting, correlation functions were sampled at 25.6kHz and an average of 55 realisations of 2048 points was used to determine the auto and cross correlation functions. The auto and cross spectra were calculated and from these the coherence and phase of the data was obtained.

5. Results

A comparison of the estimated autospectra with the original spectra is shown in figure 2. The sample-and-hold (S&H) and slot correlation (S-C) results for the spectrum of \( U_1(t) \) are shown in figures 2(a) & (b). For this, the mean data rate was 500 Hz and it can readily be seen that the spectrum is well estimated to over 1000Hz for both methods. For \( U_2(t) \), the data was sampled a three mean rates and the results, shown in figures 2(c) & (d), again show excellent estimates to well above the mean data rates. The results for the case of similar data (i.e. \( \gamma^2=1.0 \) & \( \phi=0 \)) are shown on figure 3. From this, it can be seen that the coherence estimates for the mean data rates of 500\& 250Hz (green) are good up to nearly 500 Hz and above this, the estimates are poor. For mean data rates of 500Hz, the estimates are good up to 800 Hz and for rates of 500 & 1000Hz, the values are fine up to nearly 1000Hz. The ranges in which good estimates of phase are obtained are the same as those for the coherence. As the coherence of the two signals drops, the ranges over which good estimates are obtained diminishes. For the data sets at 500Hz, from figure 4, the range for coherence of 0.8 is still close to 800Hz but for coherence of 0.5 in figure 5, this is reduced to 500 Hz for the coherence although the phase values are still good up to 800Hz. This is also the case for the coherence at 0.3 as seen in figure 6. The variance of the estimates increases as the coherence reduces as can clearly be seen from figures 3, 4, 5 & 6. For the slot correlation (S-C) results, the coherence estimates are generally good for the same range of frequencies as the S&H results. However, for the phase, the upper frequency limit of the estimates is lower than that for the S&H. For coherences less than 1.0, the values are overestimated above a frequency of 500Hz compared with 700Hz for the S&H. It seems that the limitation of the S-C for estimates of phase is the lower mean data rate.

6. Conclusions

A procedure by which the sample-and-hold reconstruction technique can be used for the determination of cross power spectra of turbulence using LDV data has been given. A series of simulations have been performed to implement the procedures and the results have been compared with those obtained from the slot correlation method. The effect of mean data rates and the coherence and phase of the data on the estimates using the procedures have been examined and the following conclusions can be drawn.

- The procedures proposed produce excellent results up to and above the mean sample rate of the data.
- The results for coherence are equivalent to those obtained from the slot correlation procedures whereas those for phase are better.
- The computational time required for the S&H approach is significantly less than that for S-C.
- The effect of extraneous noise on the estimates needs to be examined.
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References
Figure 2: Comparison of original & estimated auto-spectra
Figure 3: Comparison of coherence & chase estimates ($\gamma^2=1.0$)
Figure 4: Comparison of coherence & phase estimates ($\gamma^2=0.8$)
Figure 5: Comparison of coherence & phase estimates ($\gamma^2=0.5$)
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<td>(green–250Hz: blue-500Hz: red-1000Hz)</td>
<td>(d) S-C Phase ($\phi=2\pi\tau_3$)</td>
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**Figure 6**: Comparison of coherence & phase estimates ($\gamma^2=0.5$)