Anomalous diffusion in a time-periodic, two-dimensional flow

by

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ABSTRACT

Anomalous diffusion is experimentally studied in a time-periodic, two-dimensional flow. In particular superdiffusive behavior due to the synchronization between different characteristic frequencies of the investigated flow is emphasized. The flow is generated by applying an electromagnetic forcing on a thin layer of an electrolyte solution as showed in Figure 1. This forcing is obtained by driving two electric current *I* and *I'* between two couples of electrodes placed on opposite sides of the cell. Moreover a metallic plate with opposite-signed small permanent Neodymium magnets is placed just beneath the bottom of the cell to create a spatially periodic magnetic field. A time dependent flow can be easily obtained by changing the time dependence of the electric fields. Our aim is to investigate transport phenomena in a Lagrangian framework. As a matter of fact, the interest in studying the properties of Lagrangian trajectories is in the characterization of the mixing properties of passive tracers. One of the most important questions to investigate in transport problems is whether the only knowledge of the structure of the velocity field allows the evaluation of passive tracer dispersion properties. This reveals essentially in understanding the nature of transport *i.e.* whether it is normal or anomalous and in the evaluation of the diffusion coefficient in both cases.
Particle Tracking Velocimetry (PTV) is used to measure the flow field. This technique is the most suitable for studying dispersion phenomena in a Lagrangian framework allowing the direct evaluation of particle displacements and related quantities. Moreover, due to the characteristics of the analyzed flow and to the improvement of the tracking procedure, we have been able to track a great number of particles for time intervals greater than the characteristic time-scales of the flow (Figure 2). By interpolating Lagrangian data over a regular grid, a flow description in an Eulerian framework is gained too.

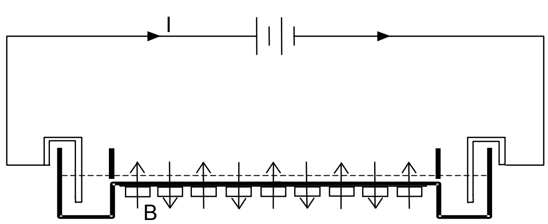


Figure 1 Experimental set-up.

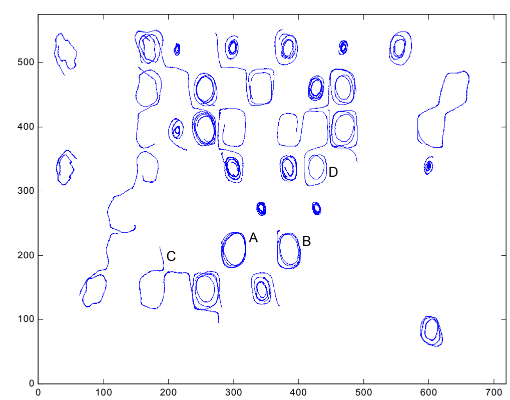


Figure 2 Example of reconstructed trajectories over 2280 frames.

1 INTRODUCTION

Transport of scalar quantities passively advected by a given velocity field, is a problem of great concern both in theoretical and applied research fields *i.e.* porous media, geophysics, astrophysics and chemical engineering (*Moffat*, 1983). As a matter of fact, the combination of advection and molecular diffusivity effects can reveals in nontrivial behaviours even in presence of simple laminar velocity fields. In this paper, we will focus on a particular aspect leading to superdiffusion.

From a general point of view, if the equation of particles motion are considered in a Lagrangian framework, it is easy to prove the possible occurrence of chaotic solutions *i.e.* particles trajectories are practically indistinguishable from those obtained in complex flows (*Ottino*, 1989; *Crisanti et al.*, 1991). When the onset of this mechanism occurs, the tracer is stirred more efficiently than when the only effects of molecular diffusivity are considered; as a consequence an efficient mixing is obtained.

The investigation of passive tracers diffusion is classically carried out considering long-time, large-distance *i.e.* an asymptotic behaviour. As a consequence, this approach works when times much larger than the characteristic time of the velocity field are considered and particles have 'sampled' the whole different flow regimes in the system. The above scenario leads to the following particles behaviour:

$$\left\langle (x(t)-x(0))^2 \right\rangle \approx t \qquad (1)$$

which is characterised by a diffusion coefficient D generally larger than the molecular diffusion coefficient D_0 (*Mazzino*, 1997). We refer to this behaviour as to diffusive behaviour. In this framework, given the velocity field, the evaluation of D can be successfully carried out for many classes of flows by using some standard mathematical tools. For a review on this subject see *Majda & Kramer* (1999).

Nevertheless, there are some important physical systems in which particles doesn't show a diffusive behavior in the limit of very long time but follow a different law *i.e.*:

$$\left\langle (\mathrm{x(t)}-\mathrm{x(0)})^2 \right\rangle \approx \mathrm{t}^{2\upsilon} \quad ; \quad \upsilon \neq 1/2 \qquad (2)$$

This scenario corresponds to an anomalous diffusive behaviour. In particular one speaks of subdiffusive behaviour when $\nu<1/2$ and superdiffusive behaviour when $\nu>1/2$. Moreover, many systems of geophysical concern, display a long time diffusive behavior (1) but a transient anomalous diffusive regime (2) which can also be very long with respect to the characteristic time scales of the flow. It is proved that if the velocity field is incompressible and the molecular diffusivity is non-zero, either standard diffusion or superdiffusion can take place (*Castiglione et al.*, 2001). As a consequence, we will focus on this aspect of anomalous diffusive behavior. In a given system, superdiffusion occurs when one of the two following conditions fails (*Vergassola*, 1998):

- finite variance of the velocity;
- fast enough decay of the autocorrelation function of Lagrangian velocities.

The abovementioned conditions corresponds to the hypothesis of the central limit theorem; as a matter of fact at the basis of all standard diffusive processes there is a *random walk* process and quantities evolving on times-scales much larger than the mixing time scale will be governed by equation (1). When some, or even all, the hypothesis of the central limit theorem fail, an anomalous diffusion process arises. This reveals essentially in a lack of a sharp scale separation between microscopic and macroscopic time scales and the corresponding dynamics. It has to be observed that while a violation of the first condition corresponds to quite unphysical situations, systems in which the second condition is violated are much more encountered in physical applications. Several authors (*Castiglione et al.*, 2001) have shown how, due to the divergence of the decorrelation time between micro and macro dynamics (i.e. to the failure of the 2^{Th} condition), a superdiffusive behavior arises in quasi-geostrophic, planetary flows leading particles to jump for very long time in the same direction.

Moreover, two recent papers (*Vulpiani,* 1998; *Solomon*, 2001) show how, also in very simple two-dimensional, periodic in time and space flows, an anomalous diffusive behaviour (2) can appear. In particular when molecular diffusivity is nonzero, a transient anomalous behaviour occurs and it is stressed by the dependence of the diffusion coefficient from the frequency of the flow characterized by resonance peaks. In the limit of zero molecular diffusivity, in a narrow window in correspondence of these peaks, the anomalous behaviour turns to be an asymptotic property and the exponent ν in equation (2) is not a constant. With the idea of test these results via experiments, we perform an experimental study of transport in an electromagnetically forced time and space periodic two-dimensional flow. The flow is generated by applying an electromagnetic forcing on a thin layer of an electrolyte solution and reveals in a square grid of alternating vortices. Time dependence can be easily obtained by changing the time dependence of the electric fields. In particular, considering certain values of the imposed oscillation frequencies, particles can display very long jump. Particle Tracking Velocimetry (PTV) is used to measure the flow field. This technique is the most suitable for studying dispersion phenomena in a Lagrangian framework allowing the direct evaluation of particle displacements and related quantities.

2 EXPERIMENTAL SET-UP AND MEASURING TECHNIQUE

The experimental set-up (Figure 3) consists of a square cell $50\times50 cm^2$, 5 *cm* high, made of plexiglass and partially filled with an electrolyte solution. The flow is generated by driving two electric current *I* and *I'* between two couples of electrodes placed on opposite sides of the cell. Two orthogonal easily adjustable electric fields are then obtained. In order to reduce impurities contamination caused by electrolysis processes, the electrodes have been placed in two external lateral reservoirs connected from the bottom to the cell.
A metallic plate with opposite-signed small permanent Neodymium magnets is placed just beneath the bottom of the cell to create a spatially periodic magnetic field (max amplitude ~ 0.5 *Tesla*). The combination of the electric currents and the vertical magnetic field causes the force that drives the flow. In order to increase the stability of the flow and to reduce Reynolds number, we mix the electrolyte (H_2O+NaCl) with glycerine (33% solution).

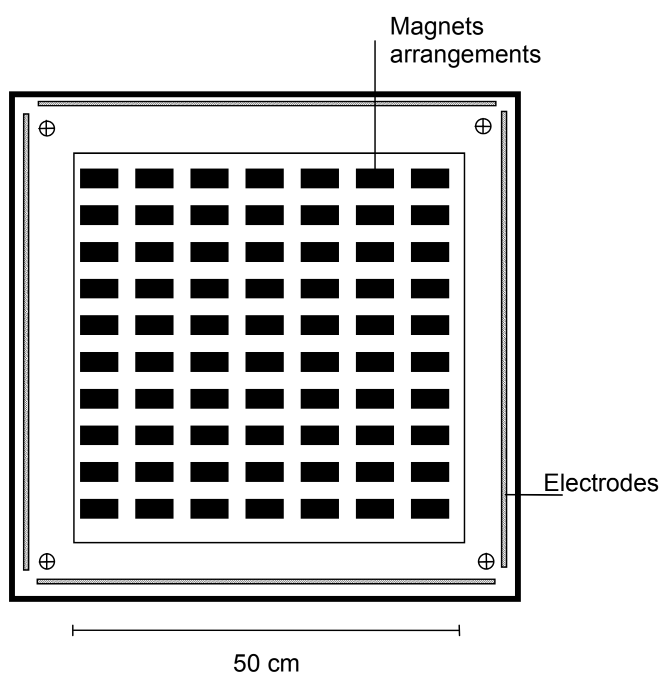


Figure 3 Top view of the experimental apparatus.

Several experiments using similar arrangements have been carried out by *Cardoso* & *Tabeling* (1988) aimed to the characterization of dispersion in a linear array of vortices and by *William et. al* (1997) aimed to the study of mixing properties of laminar flow. A large amount of experiments have been focused on the study of two-dimensional turbulence (*Paret & Tabeling*, 1997). In both cases, the magnets are arranged in a regular pattern generating a periodic forcing. The control parameter is the current *I*. For small and constant *I*, the flow displays an array of stationary vortices. For larger value of *I*, vortices become unstable and a non-stationary flow is observed. Turbulence is generated with high and time-dependent current *I*.

In our arrangement, if *I* is small and *I'* =0, the flow is stationary, the vortex pattern is very stable. Time dependence is easily obtained by sinusoidally inverting the polarity of *I'* with a frequency *f* chosen according to the time scales of the flow. In fact by modulating *I'*, we obtain a periodic stream function and under this condition it is possible to observe diffusion driven by chaotic advection (Figure 6b, 6c, 6d).

A Lagrangian description of the flow is obtained by using the PTV technique. The fluid is seeded with 250 μm styrene particles. The test section is illuminated by two 500W lamps. Images are acquired by a 3-CCD camera (1/25s shutter) placed orthogonally to the upper surface of the tank.

The procedure used for detecting trajectories mainly consists of four steps (*Cenedese & Querzoli*, 2000):

- image acquisition (RGB 24 bit, PAL Standard in this context);
- best buffer identification and choice of the threshold value;
- centroids identification;
- trajectories reconstruction and velocity evaluation.

In the first two steps, acquired images are digitised and pre-processed in order to improve both the buffer identification and the choice of the threshold value. As a matter of fact, images usually available for PTV analysis, result as the superposition of lighted spots over a background. Light reflections and refractions add noise both as localized spots and higher background level, depending on the accuracy of the experimental procedures and of images acquisitions. The improvement of the centroids identification procedure strongly influence the persistence of a single centroid characteristics in a sequence of images (mostly in 2D flows) and allows a better set of data as input for tracking algorithm.

The aim is then to have at each time a number of alive trajectories of the same order of magnitude of the identified centroids at the same time. Before thresholding, a background subtraction is performed on images. Background at a given location is computed as the average of the grey levels in a square neighborhood centered on the considered pixel. The main effect of the background removal is to account for the variation of the black level over the image, due to the non-uniform illumination of the investigated area. In Figures 4, 5 the effects of the pre-processing are showed: in particular, Figure 4 represent a raw image while in Figure 5 the same image after pre-processing procedure has been plotted. In order to enhance the contrast, we plot the negative of images so that dark spots correspond to particles while white pixels correspond to the background.

After thresholding, the digitized frames are reduced to Boolean images so that the non-zero values can be associated to the particle images while zeros can be associated to the background. Each Boolean image is then labeled for identifying sets of connected non-zero pixels which are good candidate particle images. Finally these sets are selected according a minimum and a maximum limit value on their areas and stored within temporal information.

A tracking algorithm allows trajectories reconstruction by detecting, positioning and tracking individual particles images over a set of acquired frames. In particular the version of the algorithm we used, implies sub-pixel accuracy in measuring particle displacements. Trajectories were recognised evaluating the position of particles on subsequent frames on the basis of two limiting criteria: maximum distance between particles (i.e. maximum velocity) and maximum difference between successive particle displacements (i.e. maximum acceleration). Moreover, when more than one particle satisfied the latter criterion, the ambiguity is solved by choosing the one corresponding to the minimum acceleration.

Our goal is to follow particle as long as possible *i.e.* to obtain very long trajectories in order to evaluate Lagrangian statistics on particle displacements and quantify the dispersion phenomena occurring in the tank.

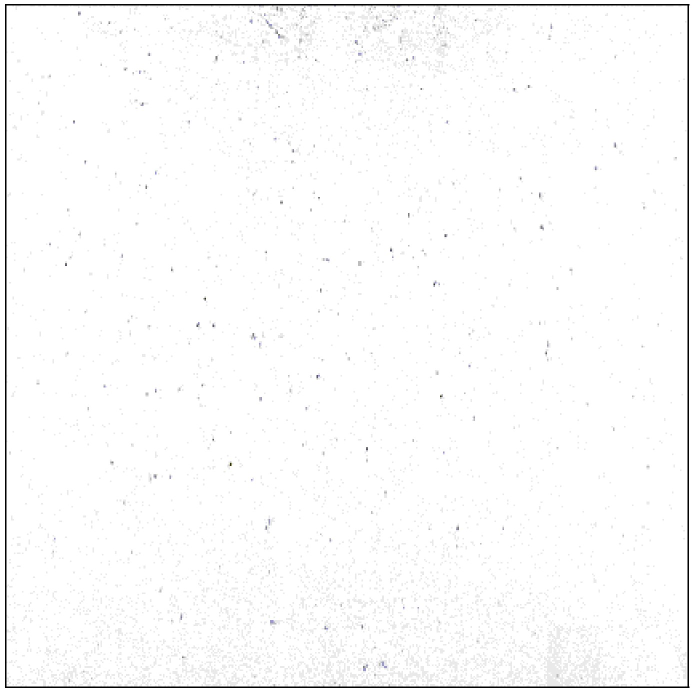


Figure 4 Example of a raw image.

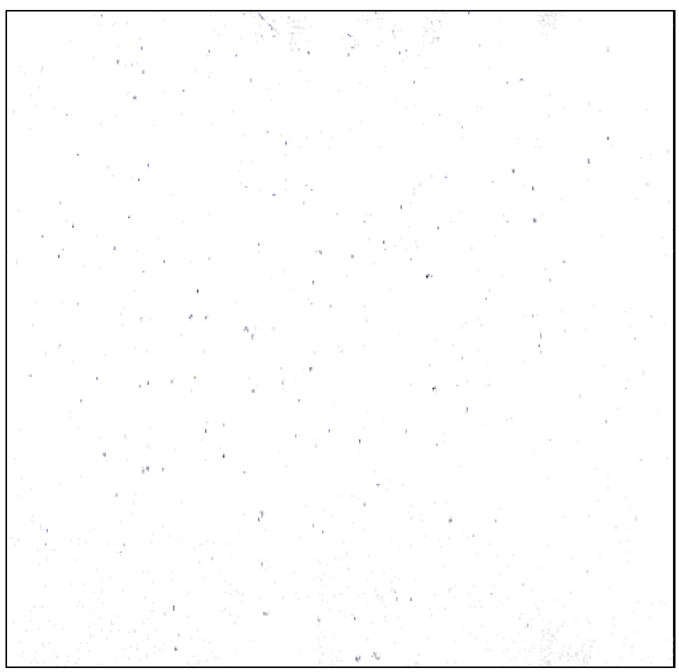


Figure 5 Effect of background subtraction on the same image.

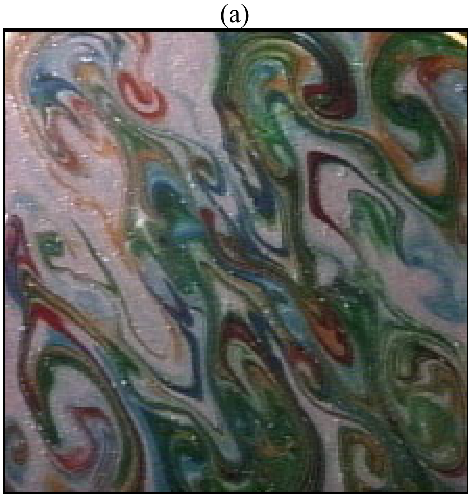
3 EXPERIMENTAL RESULT

3.1 Qualitative results

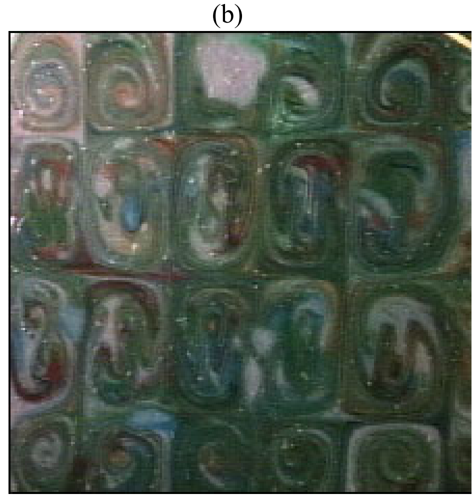
The different flow configurations described in the previous section, have been qualitatively analyzed by injecting colored dye in the flow domain. Figure 6 shows the flow structure in stationary (a) and time periodic regimes (b), (c), (d). The figure clearly shows how particle spreading can be enhanced when, due to the periodicity of the streamfunction obtained by tuning I', the flow separatrices are tilted (b) and tracer particles can jump from one cell to the others. The synchronization between the circulation in the cells and their oscillation is in fact a very efficient way for particles to jump from cell to cell. As already discussed, under chaotic conditions mixing is more efficient (c, d).

(a)

(b)



(c)

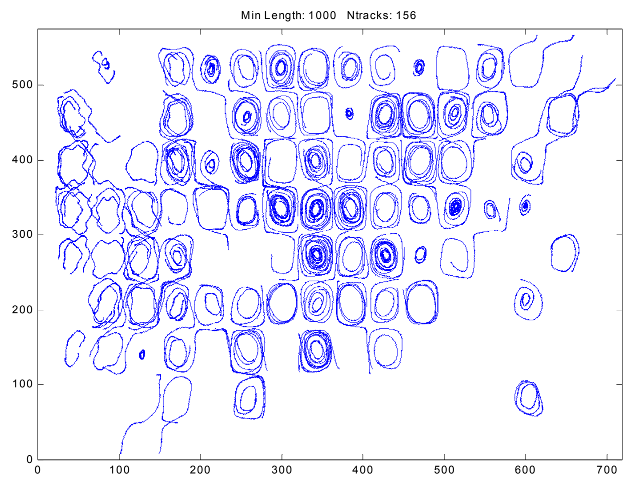


(d)

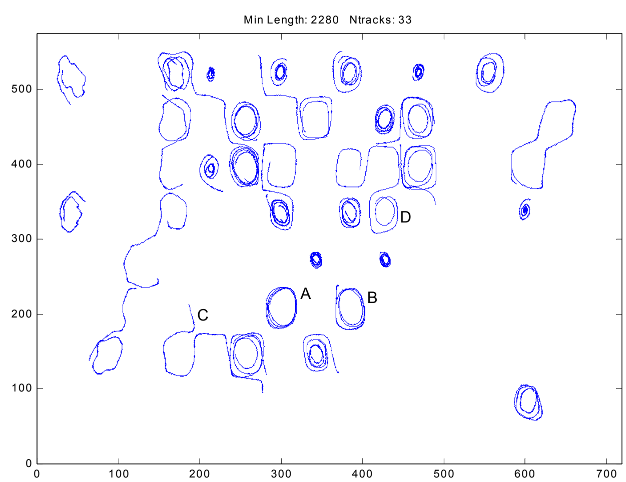
Figure 6. Flow structure in stationary (a) and time periodic regimes (b), (c), (d) corresponding to following time instants. Enhanced Mixing after periodic perturbation is evident (c), (d).

3.2 Quantitaive results

Figures 7a, 7b show recognised trajectories respectively during 1000 frames (52s) and 2280 frames (120s) of an acquisition. The experimental parameters were: *V*=4 *Volts*, *V'*=2 *Volts* (*V*, *V'* respectively represent the voltages corresponding to *I*, *I'*), *f*=0.09 *Hz*. These figures clearly show the sensitivity of particle trajectories to initial conditions: as shown in Figure 7b, some particles (i.e. A, B) remain trapped in the core of vortices while other particles (i.e. C, D) jump among different cells. Figure 8 shows some examples of selected trajectories in correspondence of the resonance frequency: the particle jumps in the adjacent cells are evident.



(a)



(b)

Figure 7. Reconstructed trajectories after 52*s* (a) e 120*s* (b).

From particle trajectories, Lagrangian velocity fields are evaluated trivially by dividing the particle displacements by the time interval between frames. By an interpolation procedure, the Eulerian instantaneous velocity fields are obtained too. Figure 9 shows the mean velocity field obtained in the same acquisition considering a 65×70 regular grid. Although PTV allows for the evaluation of velocity vectors with high local accuracy and assures a statistical independence of data, this procedure of interpolation should be carefully carried out in order to avoid errors (Adrian, 1991). By processing Eulerian velocity fields, it is possibile to evaluate instantaneous vorticity distributions. In Figure 10 the iso-surfaces of instantaneous vorticity are plotted. The flow pattern corresponding to the onset of synchronism mechanism between rotation and oscillation frequencies is stressed and channels of preferential motion for particles are evident.

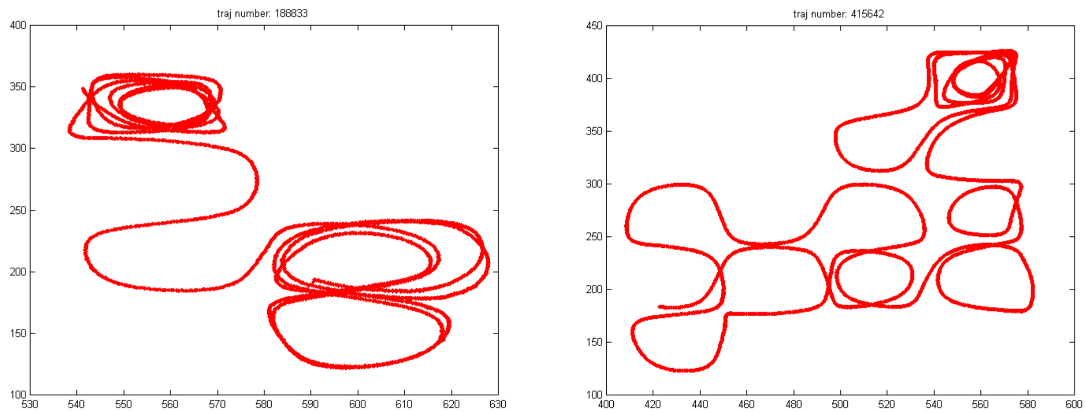


Figure 8 Trajectory evolution in correspondence of resonance frequency

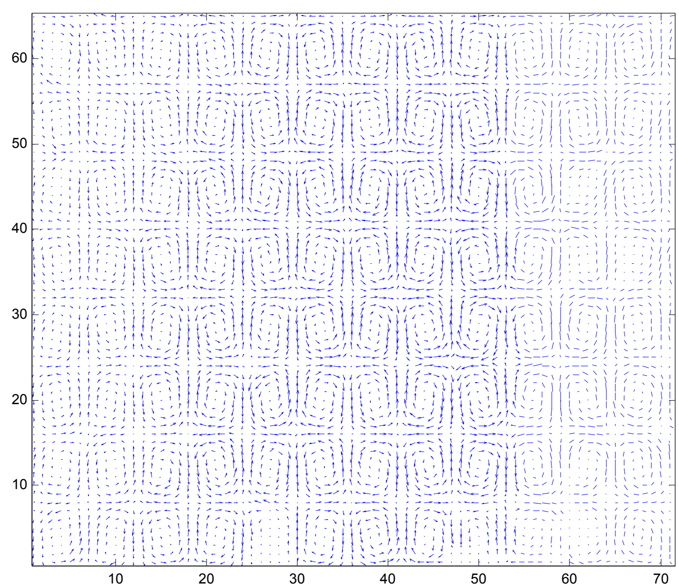


Figure 9 Mean velocity field

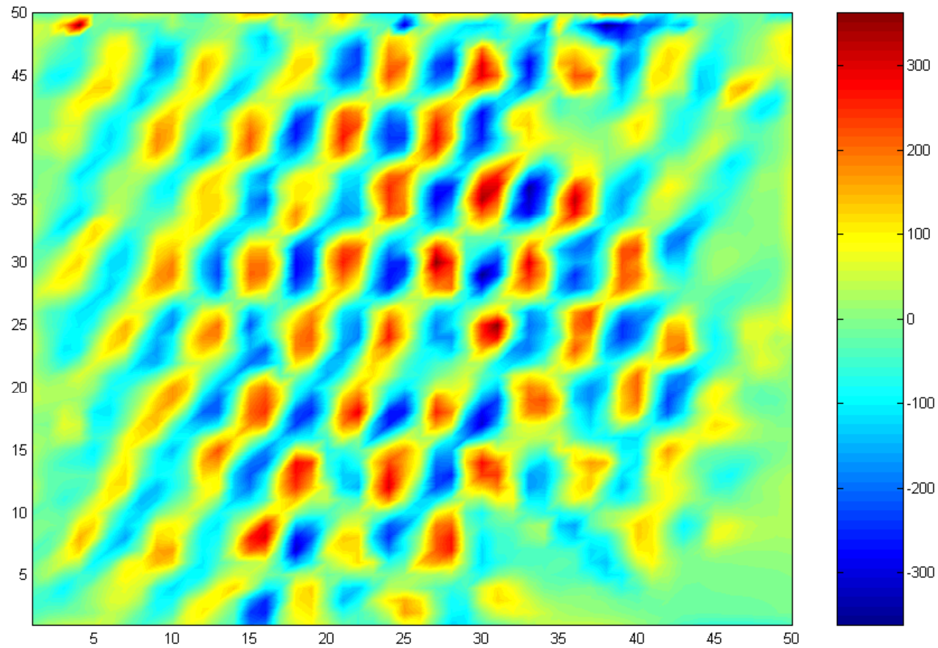


Figure 10 Instantaneous vorticity distribution (time=frames).

In order to characterize dispersion phenomena, we then evaluate Lagrangian statistics on particle displacements. In particular, considering the displacement vector $\boldsymbol{r=(x(t)- x(0))}$, we plot $<r(t)^2>$ vs. t (see Figure 11). In the correspondent experimental acquisition, the ratio between the rotation frequency $\omega/2\Pi$ and the rolls oscillation frequency f is close to unit. As a matter of fact, even if at highest frequencies we point out an impact of the lack of statistics, we observe a non linear trend indicating a superdiffusive behaviour. Moreover the strongly oscillating behaviour observed at low frequencies corresponds to opening and closing of the aforementioned channels for particle motion at the frequency f.

4 CONCLUSIONS

In this work we present several qualitative and quantitative experimental result corresponding to scalar transport in a two-dimensional, space and time periodic flow. The behaviour of diffusion in the analysed context, is expected to be anomalous-type. In particular, due to the possibility of the onset of resonance mechanisms between some characteristic frequency of the flow, a superdiffusive behaviour is expected.
Both the utilized tracking procedure and the particular experimental configuration designed, allow to reconstruct a considerable amount of very long trajectories i.e. to follow several particles for time longer than the characteristic time-scale of the flow. These dataset allow evaluating significant statistics on long time (*i.e.* asymptotic) behaviour too. The obtained results show the possibility to easily reproduce, and consequently analyse, the conditions leading to a superdiffusive behaviuour.
Aim of future works will be to reconstruct the experimental behaviour of the diffusion coefficient versus the ratio between the characteristic frequencies, identify the peaks of this curve and perform more measurements in this narrower neighbourhood. From a general point of view, we aim at testing via experiments, non-asymptotic approach and theories in order to quantify diffusion when the classical approach fails *i.e.* when it doesn't give any relevant information on the acting mechanisms (*Castiglione et. al*, 2001).

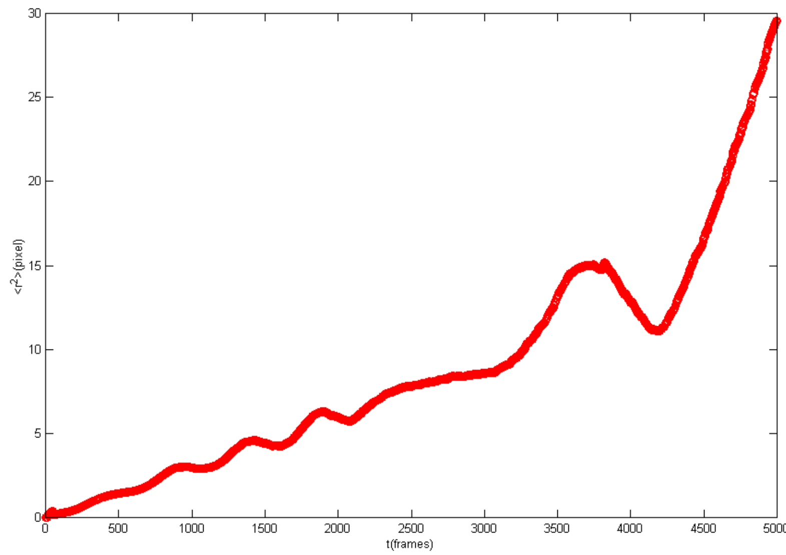


Figure 11. $<r(t)^2>$ vs. time.

ACKNOWLEDGMENTS

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